Exact Solutions Of Some Nonlinear complex fractional Partial Differential Equations

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ABSTRACT: *In this paper, we used the sub-equation method for solving the nonlinear complex fractional Schrödinger equation, the nonlinear complex fractional Kundu-Eckhaus, and nonlinear complex fractional generalized-Zakharov equations in the sense oftheJumarie's modified Riemann-Liouvillederivative. Whit the aid of the mathematical software Maple, some exact solutions for these equations are successfully.*

KEYWORDS: Sub-equation method, exact solutions ofsomenonlinearcomplex fractionalpartial differential equations.

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order, have become a focus of many studies due to their frequent appearance in various applications, for example, many important phenomena in electromagnetism, acoustics, viscoelasticity, electrochemistry [1] and material science are well described by fractional differential equations [2].To find the explicit solutions of linear and nonlinear fractional differential equations, many powerful methods have been used such as: the fractional sub-equation method [3–5], the (G′/G)-expansion method [6- 8], the Exp-function method [9-12], the tanh-coth method [13-14], the Adomian's decomposition method [15], thecosine-function method [16], the extended multiple Riccati equations expansion method [17] and the first integral method [18], and so on.In this paper we have considered the following complexNFCPDEs:

(**1**)Thenonlinear complex fractional Schrödinger equation

$$\frac{\partial^\alpha q}{\partial t^\alpha} + i q_{xx} + (|q|^2 q)_x = 0 \quad (1)$$

(**2**) Thenonlinear complex fractional Kundu-Eckhausequation

$$i\frac{\partial^\alpha q}{\partial t^\alpha} + q_{xx} - 2\gamma|q|^2 q + \delta^2|q|^4 q + 2i\delta(|q|^2)_x q = 0 \quad (2)$$

(**3**)Thenonlinear complex fractional generalized-Zakharov equations

$$\begin{cases} i\frac{\partial^\alpha q}{\partial t^\alpha} + q_{xx} - 2\alpha|q|^2 q + qr = 0 \\ \frac{\partial^{2\alpha} r}{\partial t^{2\alpha}} - r_{xx} + (|q|^2)_{xx} = 0 \end{cases} \quad (3)$$

Where q is a function in two independent variables (x,t), $0 < \alpha < 1$, the timederivativeis the Jumarie'smodified Riemann-Liouville derivative**.**This paper is arranged as follows: In Section 2, we present concepts that we need them to convert the proposed (NFCPDE) into a (ODE). In Section 3, we give the description for main steps of the sub-function method. In Section 4, we apply this method to finding exact solutions for the equationswhich we stated above

2.Preliminaries

In this section we list the definition and some important properties of Jumarie's modified Riemann-Liouville derivatives of order α as follows:

Definition 2.1 let f(t) be a continuous real (but not necessarily differentiable) functionand let h > 0 denote a constant discretization. Then the Jumarie's modified Riemann-Liouville derivative is defined as [19- 21]:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^t (t-u)^{-\alpha-1}(f(u)-f(0))du, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-u)^{-\alpha}(f(u)-f(0))du, & 0 < \alpha < 1 \\ (f^n(t))^{\alpha-n}, & n \le \alpha < n+1 \end{cases} \quad (4)$$

Where

$$D_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{k=0}^{\infty}(k)^{-1} f[x+(\alpha-k)h] \quad (5)$$

In addition, some properties for the proposed modified Riemann-Liouville derivatives as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha} \quad (6)$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t) \quad (7)$$

$$D_t^\alpha f[g(t)] = f_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha \quad (8)$$

This is the direct consequence of the following equation:[22]

$$D_t^{\alpha} f(t) = \Gamma(1+\alpha) Df(t) \quad (9)$$

which holds for non-differentiable functions

3. Description of the sub-equation method

In this section we gave a brief description for the main steps of the sub-function method. For that, consider a nonlinear partial evolution equation

$$P(q, q_t, q_x, q_{tt}, q_{xt}, \dots) = 0 \quad (10)$$

Where P a polynomial in q and its partial derivatives .In order to solve it using sub-equation method, we give the following main steps [23]

Step1. Using the wave transformation

$$q(x,t) = u(\xi), \xi = x - ct \quad (11)$$

Where c is an arbitrary constant to be determined.

From Eq. (10) and Eq. (11) we have the following ODE

$$R(u, u', u'', \dots) = 0 \quad (12)$$

Where Ris a polynomial of u and its derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

Step2. We suppose that Eq. (12) has the formal solution:

$$u(\xi) = \sum_{i=o}^{M} a_i \varphi^i \quad (13)$$

Where M is a positive integer, which can be find by balancing the highest order derivative term with the highest nonlinear terms in Eq. (12), and a_i $(i = 0,1,2,\dots,M)$ all are constants to be determined later, and $\varphi(\xi) = \varphi$ satisfies the Riccati equation.

$$\varphi'(\xi) = \sigma + \varphi^2(\xi) \quad (14)$$

Where σ is a constant.

Step3. We list the exact solutions of Eq. (14) as follows, which is known to us.

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi), & \sigma < 0 \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi), & \sigma<0 \\ \sqrt{\sigma}\tan(\sqrt{-\sigma}\xi), & \sigma>0 \\ -\sqrt{\sigma}\cot(\sqrt{-\sigma}\xi), & \sigma>0 \\ \frac{-1}{\xi+\omega}, & \sigma=0 \end{cases} \quad (15)$$

Where ω is a constant.

4. Applications

In this section, we apply the sub-functionmethod for solvingthe nonlinear complexfractional partial differential equations which we mentioned in the section (1).

Example 4.1 Consider thenonlinearcomplexfractional Schrödinger equation

$$\frac{\partial^{\alpha} q}{\partial t^{\alpha}} + iq_{xx} + (|q|^2 q)_x = 0, 0 < \alpha < 1. \quad (16)$$

In [24], the authors solved Eq. (16) by a proposed $\left(\frac{G'}{G}\right)$ expansion method in the classical caseand established some exact solutions for them. Now we will apply the described method above to Eqs. (16). To begin with, we suppose

$$q(x,t) = u(\xi)e^{i\eta}, \xi = ik\left(x + \frac{2\lambda t^{\alpha}}{\Gamma(1+\alpha)}\right), \ \eta = \left(\lambda x + \mu\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \quad (17)$$

Where,k,λ, and μ, are arbitrary constants to be determined later. Based on this and by help of Eqs. (6-8) we can easily drive

$$\frac{\partial^{\alpha} q}{\partial t^{\alpha}} = i(\mu u + 2\lambda k u')e^{i\eta} \quad (18)$$

$$q_{xx} = -(\lambda^2 u + 2\lambda k u' + k^2 u'')\, e^{i\eta} \quad (19)$$

$$(|q|^2 q)_x = i(\lambda u^3 + 3ku^2 u'')\, e^{i\eta} \quad (20)$$

Substituting Eqs. $(18-20)$into Eq. (16) we get the following ODE

$$(\mu - \lambda^2)u - k^2 u'' + \lambda u^3 + 3ku^2 u' = 0 \quad (21)$$

Balancing u'' with $u^2 u'$ in Eq. (21) we obtain$M = \frac{1}{2}$. To obtain an analytic solution, M should be an integer; this requires use of the transformation [25]

$$u(\xi) = (v(\xi))^{\frac{1}{2}} \quad (22)$$

Substituting Eq. (22) into Eq. (21) we get

$$4(\mu - \lambda^2)v^2 + k^2(v')^2 - 2k^2 vv'' + 4\lambda v^3 + 6kv^2 v' = 0 \quad (23)$$

Balancing $v^2v'$with v^3we in Eq.(23) obtain $M = 1$.Thus Eq. (13) becomes

$$v(\xi) = a_0 + a_1\varphi(\xi)(24)$$

Substituting Eq. (24) into Eq. (23) along with Eq. (14) and then setting the coefficients of $\varphi^i(i = 0,1,2,\dots,M)$ to zero, we have the set of algebraic equations about k, λ, μ, a_0 and a_1 as follows:

$$\varphi^0: 4(\mu - \lambda^2)a_0^2 + 6ka_1a_0^2\sigma + 4\lambda a_0^3 + k^2a_1^2\sigma^2 = 0$$

$$\varphi^1: 12ka_0a_1^2\sigma + 12\lambda a_0^2a_1 - 4k^2a_0a_1\sigma + 8(\mu - \lambda^2)a_0a_1 = 0$$

$$\varphi^2: 2k^2a_1^2\sigma - 12\lambda a_0a_1^2 - 6ka_0^2a_1 - 6ka_1^3\sigma - 4(\mu - \lambda^2) = 0$$

$$\varphi^3: 4\lambda a_1^3 + 12ka_0a_1^2 - 4k^2a_0a_1 = 0$$

$$\varphi^3: 3k^2a_1^2 - 6ka_1^3 = 0:$$

Solving these equations with the aid of Maple we obtain

$$a_0 = \frac{-\lambda}{2}, a_1 = \frac{\pm\lambda i}{2\sqrt{\sigma}}, k = \frac{\pm\lambda i}{\sqrt{\sigma}}, \mu = 2\lambda^2(25)$$

From (25) and Eq. (24) along with Eq. (15) we can find the following exact solutions of Eq. (16) as follows:

$$q_1(x,t) = \pm\frac{\sqrt{2\lambda}}{2}i\sqrt{1 + i\tan\left(\lambda x + \frac{2\lambda^2}{\Gamma(1+\alpha)}t^\alpha\right)}\, e^{i\left(\lambda x + \frac{2\lambda^2}{\Gamma(1+\alpha)}t^\alpha\right)}(26)$$

$$q_2(x,t) = \pm\frac{\sqrt{2\lambda}}{2}\sqrt{-1 + i\cot\left(\lambda x + \frac{2\lambda^2}{\Gamma(1+\alpha)}t^\alpha\right)}\, e^{i\left(\lambda x + \frac{2\lambda^2}{\Gamma(1+\alpha)}t^\alpha\right)}(27)$$

Example 4.2 Consider thenonlinear complex fractional Kundu-Eckhaus equation

$$i\frac{\partial^\alpha q}{\partial t^\alpha} + q_{xx} - 2\gamma|q|^2q + \delta^2|q|^4q + 2i\delta(|q|^2)_xq = 0$$
(28)

In [24]again, the authors solved Eq. (28) by a proposed $\left(\frac{G'}{G}\right)$ expansion method in the classical case and established some exact solutions for them. Now we will apply the described method above to Eqs. (28). To begin with, we suppose

$$q(x,t) = u(\xi)e^{i\eta}, \xi = ik\left(x - \frac{2\lambda t^\alpha}{\Gamma(1+\alpha)}\right),\ \eta = \left(\lambda x + \mu\frac{t^\alpha}{\Gamma(1+\alpha)}\right)(29)$$

Where,k, λ, and μ, are arbitrary constants to be determined later. From Eq. (29)and Eqs. (6-8) yields

$$\frac{\partial^\alpha q}{\partial t^\alpha} = i(\mu u - 2\lambda k u')e^{i\eta}(30)$$

$$q_{xx} = -(\lambda^2u + 2\lambda k u' + k^2u'')\, e^{i\eta}(31)$$

$$(|q|^2)_xq = 2iku^2u'\, e^{i\eta}(32)$$

Substituting Eqs. (30 − 32)into Eq. (28) we get the following ODE

$$-(\mu + \lambda^2)u - k^2u'' - 2\gamma u^3 + \delta^2u^5 - 4\delta k u^2u' = 0(33)$$

Balancing u'' with u^5 in Eq. (33) we obtain $M = \frac{1}{2}$. To obtain an analytic solution, M should be an integer; this requires use Eq. (22) into Eq. (33) yields

$$-4(\mu + \lambda^2)v^2 + k^2(v')^2 - 2k^2vv'' - 8\gamma v^3 + 4\delta^2v^4 - 8k\delta v^2v' = 0(34)$$

Balancing vv'' with v^4 in Eq. (34) we obtain $M = 1$. Thus Eq. (13) becomes

$$v(\xi) = a_0 + a_1\varphi(\xi)(35)$$

Substituting Eq. (35) into Eq. (34) along with Eq. (14) and then setting the coefficients of $\varphi^i(i = 0,1,2,\dots,M)$ to zero, we have the set of algebraic equations about $k, \lambda, \mu, a_0$and a_1 as follows:

$$\varphi^0: 4(\mu + \lambda^2)a_0^2 + 8\gamma a_0^3 + 8k\delta a_0^2a_1\sigma - k^2a_1^2\sigma^2 - 4\delta^2a_0^4 = 0$$

$$\varphi^1: 24\gamma a_0^2a_1 + 4k^2a_0a_1\sigma + 8(\mu + \lambda^2)a_0a_1 - 16\delta^2a_0^3a_1 + 16k\delta a_0a_1^2\sigma = 0$$

$$\varphi^2: 8k\delta a_1^3\sigma - 24\delta^2a_0^2a_1^2 + 8k\delta a_0^2a_1 + 4(\mu + \lambda^2)a_1^2 + 2k^2a_1^2\sigma + 24\gamma a_0a_1^2 = 0$$

$$\varphi^3: 4k^2a_0a_1 + 8\gamma a_1^3 + 16k\delta a_0a_1^2 - 16\delta^2a_0a_1^3 = 0$$

$$\varphi^4: 8k\delta a_1^3 - 4\delta^2a_1^4 + 3k^2a_1^2 = 0$$

Solving these equations with the aid of Maple we obtain two sets solutionsas follows:

$$a_0 = \frac{(5+\sqrt{7})\gamma}{8\delta^2}, a_1 = \frac{(16+5\sqrt{7})\gamma i}{4\sqrt{\sigma}(5+\sqrt{7})\delta^2}, k = \frac{(16+5\sqrt{7})(\sqrt{7}-1)\gamma i}{2\sqrt{\sigma}(5+\sqrt{7})^2\delta},$$

$$\mu = -\frac{(29\gamma^2 + 4\gamma^2\sqrt{7} + 128\delta^2\lambda^2 + 40\delta^2\lambda^2\sqrt{7})}{8(16+\sqrt{7})\delta^2}.(36)$$

$$a_0 = \frac{(5-\sqrt{7})\gamma}{8\delta^2}, a_1 = \frac{(-16+5\sqrt{7})\gamma i}{4\sqrt{\sigma}(-5+\sqrt{7})\delta^2}, k = \frac{(-16+5\sqrt{7})(\sqrt{7}+1)\gamma i}{2\sqrt{\sigma}\left(-5+\sqrt{7}\right)^2\delta},$$

$$\mu = -\frac{(-29\gamma^2+4\gamma^2\sqrt{7}-128\delta^2\lambda^2+40\delta^2\lambda^2\sqrt{7})}{8(-16+\sqrt{7})\delta^2}. (37)$$

Where k and λ are arbitrary constants.

From (36), and Eq. (35) along with Eq. (15) we get the following exact solutions of Eq. (28) as follows:

$$q_1(x,t) = \rho\sqrt{1+i\tan\left(\frac{\gamma(16+5\sqrt{7})(1-\sqrt{7})\left(x-\frac{2\lambda}{\Gamma(1+\alpha)}t^\alpha\right)}{\left(5+\sqrt{7}\right)^2\delta}\right)} (38)$$

$$q_2(x,t) = i\rho\sqrt{-1+i\cot\left(\frac{\gamma(16+5\sqrt{7})(1-\sqrt{7})\left(x-\frac{2\lambda}{\Gamma(1+\alpha)}t^\alpha\right)}{\left(5+\sqrt{7}\right)^2\delta}\right)} (39)$$

Where

$$\mu = -\frac{(29\gamma^2+4\gamma^2\sqrt{7}+128\delta^2\lambda^2+40\delta^2\lambda^2\sqrt{7})}{8(16+\sqrt{7})\delta^2},$$

$$\rho = \frac{\sqrt{2\gamma(5+\sqrt{7})}}{4\delta} \times e^{i\left(\lambda x+\frac{\mu}{\Gamma(1+\alpha)}t^\alpha\right)}.$$

Similarly, from (37), we have the following exact solutions:

$$q_3(x,t) = \tau\sqrt{1+i\tan\left(\frac{\gamma(16+5\sqrt{7})(1-\sqrt{7})\left(x-\frac{2\lambda}{\Gamma(1+\alpha)}t^\alpha\right)}{\left(-5+\sqrt{7}\right)^2\delta}\right)} (40)$$

$$q_4(x,t) = i\tau\sqrt{-1+i\cot\left(\frac{\gamma(16+5\sqrt{7})(1-\sqrt{7})\left(x-\frac{2\lambda}{\Gamma(1+\alpha)}t^\alpha\right)}{\left(-5+\sqrt{7}\right)^2\delta}\right)} (41)$$

Where

$$\mu = -\frac{(-29\gamma^2+4\gamma^2\sqrt{7}-128\delta^2\lambda^2+40\delta^2\lambda^2\sqrt{7})}{8(-16+\sqrt{7})\delta^2},$$

$$\tau = \frac{\sqrt{2\gamma(5-\sqrt{7})}}{4\delta} \times e^{i\left(\lambda x+\frac{\mu}{\Gamma(1+\alpha)}t^\alpha\right)}.$$

Example 4.3 Thenonlinear complex fractional generalized-Zakharov equations

$$\begin{cases} i\frac{\partial^\alpha q}{\partial t^\alpha} + q_{xx} - 2\gamma|q|^2 q + 2qr = 0 \\ \frac{\partial^{2\alpha} r}{\partial t^{2\alpha}} - r_{xx} + (|q|^2)_{xx} = 0 \end{cases} (42)$$

In [26], the authors solved Sys. (42) by Exp-function methodin the classical case and established some exact solutions for them. Now we will apply the described method above to Sys. (51). To begin with, weAssume that

$$q(x,t) = u(x,t)e^{i\eta}, \eta = \lambda x + \frac{\mu}{\Gamma(1+\alpha)}t^\alpha (43)$$

Substituting Eq. (43) reduce the Sys. (42) into the following Sys

$$\begin{cases} i\frac{\partial^\alpha u}{\partial t^\alpha} + 2i\lambda u_x + u_{xx} - (\mu+\lambda^2)u - 2\gamma u^3 + 2ur = 0 \\ \frac{\partial^{2\alpha} r}{\partial t^{2\alpha}} - \frac{\partial^2 r}{\partial x^2} + u_{xx}^2 = 0 \end{cases} (44)$$

Suppose that

$$u(x,t) = S(\xi),\ r(x,t) = T(\xi), \xi = x - \frac{2\lambda}{\Gamma(1+\alpha)}t^\alpha (45)$$

Substituting Eqs. (45) into Sys. (44) yields

$$\begin{cases} S'' - (\mu+\lambda^2)S - 2\gamma S^3 + 2ST = 0 \\ (4\lambda^2-1)T'' + (S^2)'' = 0 \end{cases} (46)$$

Integrating the second equation of Sys. (46) and neglecting the constant of integration we find that

$$T(\xi) = \frac{S^2(\xi)}{1-4\lambda^2} (47)$$

Substituting Eq. (47) into first equation of Sys. (46) we get

$$S'' - (\mu+\lambda^2)S + \left(\frac{2}{1-4\lambda^2} - 2\gamma\right)S^3 = 0 (48)$$

Balancing $S''(\xi)$ with $s^3(\xi)$ in Eq. (48) we find that $M = 1$. Thus Eq. (13) becomes

$$S(\xi) = a_0 + a_1\varphi(\xi) (49)$$

Substituting Eq. (49) into Eq. (48) along with Eq. (14) and then setting the coefficients of $\varphi^i (i = 0,1,2,\ldots,M)$ to zero, we have the set of algebraic equations about λ, μ, a_0, a_1 as follows:

$$\phi^0: 8\gamma a_0^3\lambda^2 + 4\mu a_0\lambda^2 + 2a_0^3 - \mu a_0 + 4\lambda^4 a_0 - \lambda^2 a_0 - 2\gamma a_0^3 = 0$$

$$\phi^1: 24\gamma a_0^2 a_1\lambda^2 - \mu a_1 + 4\lambda^4 a_1 - 8a_1\sigma\lambda^2 - \lambda^2 a_1 + 2a_1\sigma$$

$$-6\gamma a_0^2 a_1 + 4\mu a_1 \lambda^2 + 6a_0^2 a_1 = 0$$

$$\varphi^2: 6\gamma a_0 a_1^2 - 6a_0 a_1^2 - 24\gamma a_0 a_1^2 \lambda^2 = 0$$

$$\varphi^3: 2a_1 + 8\gamma a_1^3 \lambda^2 - 8\lambda^2 a_1 - 2\gamma a_1^3 + 2a_1^3 = 0$$

Solving these equations with the aid of Maple we obtain

$$a_0 = 0, a_1 = \frac{\sqrt{(4\gamma\lambda^2-\gamma+1)(4\gamma\lambda^2-1)}}{4\gamma\lambda^2-\gamma+1}, \mu = 2\sigma - \lambda^2 (50)$$

Substituting Eqs. (50) into Eq. (49) along with (15) we find that

$$\begin{cases} q(x,t) = \pm \frac{\sqrt{4\lambda^2-1}\sqrt{\sigma}\tan\sqrt{\sigma}(x-2\lambda\frac{1}{\Gamma(1+\alpha)}t^\alpha)}{\sqrt{4\gamma\lambda^2-\gamma+1}} e^{i(\lambda x+\frac{(2\sigma-\lambda^2)}{\Gamma(1+\alpha)}t^\alpha)} \\ r(x,t) = -\frac{\sqrt{\sigma(4\lambda^2-1)}\tan^2\sqrt{\sigma}(2\lambda\frac{1}{\Gamma(1+\alpha)}t^\alpha-x)}{4\gamma\lambda^2-\gamma+1} \end{cases}$$

(51)

$$\begin{cases} q(x,t) = \pm \frac{\sqrt{4\lambda^2-1}\sqrt{\sigma}\cot\sqrt{\sigma}(x-2\lambda\frac{1}{\Gamma(1+\alpha)}t^\alpha)}{\sqrt{4\gamma\lambda^2-\gamma+1}} e^{i(\lambda x+\frac{(2\sigma-\lambda^2)}{\Gamma(1+\alpha)}t^\alpha)} \\ r(x,t) = -\frac{\sqrt{\sigma(4\lambda^2-1)}\tan^2\sqrt{\sigma}(2\lambda\frac{1}{\Gamma(1+\alpha)}t^\alpha-x)}{4\gamma\lambda^2-\gamma+1} \end{cases}$$

(52)

$$\sigma \neq 0 .$$

5. Conclusion

In this paper, we successfully use the sub-function method to solve fractional nonlinear partial differential equations with Jumarie's modified Riemann–Liouville derivative**.** To our knowledge, the solutions obtained in this paper have not been reported in the literature so far.

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