Some double finite integrals involving the hypergeometric function and Aleph-function of two variables

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Abstract : The aim of this document is to evaluate four finite double integrals involving the product of two hypergeometric functions and the Aleph-function of two variables defined by K. Sharma [4]. At the end of this paper , we evaluate one double integral invoving the I-function of two variables defined by C.K Sharma et al [2]

Key words : Double finite integral , hypergeometric function , Aleph-function of two variables.

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1) Introduction and preliminaries.

The Aleph-function of two variables was recently study by Kishan Sharma [4]. This function of two variables is an extension of the I-function defined by C.K. Sharma and P.L. Mishra [2] , wich itself is a generalisation of G and H-function of two variables. The double Mellin-Barnes integral occuring in this paper will be referred to as the Aleph-function of two variables throughout our present study and will be defined and represented as follows.

$$\aleph(z_1,z_2)=\aleph^{0,n:m_1,n_1:m_2,n_2}_{P_i,Q_i,\tau_i:r;P'_i,Q'_i,\tau'_i:r';P_i",Q_i",\tau_i":r"}\left(z_1,z_2\middle|\begin{matrix}\mathrm{A}(\tau_i):C(\tau_{i'});E(\tau_{i''})\\ \mathrm{B}(\tau_i):D(\tau_{i'});F(\tau_{i''})\end{matrix}\right)$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s_1,s_2)\prod_{j=1}^{2}\phi_j(s_j)z_1^{s_1}z_2^{s_2}\,\mathrm{d}s_1\mathrm{d}s_2 \tag{1.1}$$

where :

$$A(\tau_i)=(a_j,\alpha_j,A_j)_{1,n},[\tau_j(a_{ji},\alpha_{ji},A_{ji})]_{n+1,P_i}\ ;B(\tau_i)=[\tau_j(b_{ji},\beta_{ji},B_{ji})]_{1,Q_i} \tag{1.2}$$

$$C(\tau_{i'})=(c_j,\gamma_j)_{1,n_1},[\tau_j(c_{ji'},\gamma_{ji'})]_{n_1+1,P_{i'}};D(\tau_{i'})=(d_j,\delta_j)_{1,m_1},[\tau_j(d_{ji'},\delta_{ji'})]_{m_1+1,P_{i'}} \tag{1.3}$$

$$E(\tau_{i''})=(e_j,E_j)_{1,n_2},[\tau_j(e_{ji''},E_{ji''})]_{n_2+1,P_{i''}};F(\tau_{i''})=(f_j,F_j)_{1,m_2},[\tau_j(f_{ji''},F_{ji''})]_{m_2+1,Q_{i''}} \tag{1.4}$$

$\theta(s_1,s_2)$ and $\phi_j(s_j)$ are defined by K. Sharma [4].
The existence condition of (1.1) are below :

$$\Omega=\sum_{j=1}^{n_1}\alpha_j+\tau_i\sum_{j=n_1+1}^{P_i}\alpha_{ji}+\sum_{j=1}^{n_2}\gamma_j+\tau'_i\sum_{j=n_2+1}^{P_{i'}}\gamma_{ji'}-\tau_i\sum_{j=1}^{Q_i}\beta_{ji}-\sum_{j=1}^{m_2}\delta_j-\tau'_i\sum_{j=m_2+1}^{Q_{i'}}\delta_{ji'}<0 \tag{1.5}$$

$$\Delta=\sum_{j=1}^{n_1}A_j+\tau_i\sum_{j=n_1+1}^{P_i}A_{ji}+\sum_{j=1}^{n_3}E_j+\tau''_i\sum_{j=n_3+1}^{P_{i''}}E_{ji''}-\tau_i\sum_{j=1}^{Q_i}B_{ji}-\sum_{j=1}^{m_3}F_j-\tau''_i\sum_{j=m_3+1}^{Q_{i''}}F_{ji''}<0 \tag{1.10}$$

$$A=\sum_{j=1}^{n_1}\alpha_j-\tau_i\sum_{j=n_1+1}^{P_i}\alpha_{ji}+\sum_{j=1}^{n_2}\gamma_j-\tau'_i\sum_{j=n_2+1}^{P_{i'}}\gamma_{ji''}-\tau_i\sum_{j=1}^{Q_i}\beta_{ji}+\sum_{j=1}^{m_2}\delta_j-\tau'_i\sum_{j=m_2+1}^{Q_{i'}}\delta_{ji'}>0 \tag{1.11}$$

$$B =\sum_{j=1}^{n_1} A_j - \tau_i \sum_{j=n_1+1}^{P_i} A_{ji} + \sum_{j=1}^{n_3} E_j - \tau_i'' \sum_{j=n_3+1}^{P_{i''}} E_{ji''} - \tau_i \sum_{j=1}^{Q_i} B_{ji} + \sum_{j=1}^{m_3} F_j - \tau_i'' \sum_{j=m_3+1}^{Q_{i''}} F_{ji''} > 0 \quad (1.12)$$

with , $|Arg(z_1)| < \frac{\pi}{2}A, |Arg(z_2)| < \frac{\pi}{2}B$ (1.13)

Throughout the present document, we assume of that the existence and convergence conditions of the Aleph-function of two variables. For more informations , see K, Sharma [4]. We will use these following notations in this paper.

$U = A(\tau_i) : C(\tau_{i'}); E(\tau_{i''})$ and $V = B(\tau_i) : D(\tau_{i'}); F(\tau_{i''})$ (1.14)

$X = m_2, n_2; m_3, n_3$ and $Y = P_{i'}, Q_{i'}, \tau_{i'} : r'; P_{i''}, Q_{i''}, \tau_{i''} : r"$ (1.15)

2) Hypergeometric function

We have the following results , see Rathie et al [4]

$$\int_0^1 x^{\rho-1}(1-x)^{\rho}[1+ax+(1-b)]^{-2\rho-1} {}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]\mathrm{d}x$$

$$= 2^{\alpha+\beta-2\rho}\ \frac{\Gamma(\rho-\frac{\alpha}{2}-\frac{\beta}{2})\Gamma(\frac{\alpha+\beta+2}{2})\Gamma(\rho)}{(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)}$$

$$\times\left[\frac{2\rho-\alpha+\beta)\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\rho-\frac{\alpha}{2}-1)\Gamma(\rho-\frac{\beta}{2}+\frac{1}{2})} - \frac{(2\rho-\alpha+\beta)\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\rho-\frac{\alpha}{2}+1)\Gamma(\rho-\frac{\beta}{2}+\frac{1}{2})}\right] \quad (2.1)$$

Where $Re(\rho) > 0, Re(2\rho-\alpha-\beta) > 0$, a and b are constants , such the expression

$1+ax+b(1-x)$ is not zero.

$$\int_0^1 x^{\rho-1}(1-x)^{\rho}\left[1+ax+b(1-x)\right]^{-2\rho+1} {}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]\mathrm{d}x$$

$$= 2^{\alpha+\beta-2\rho-1}\frac{\Gamma(\rho-\frac{\alpha}{2}-\frac{\beta}{2}-1)\Gamma(\frac{\alpha+\beta}{2})\Gamma(\rho-1)}{(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)}$$

$$\times\left[\frac{(2\rho-\alpha+\beta-2)\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\rho-\frac{\alpha}{2})\Gamma(\rho-\frac{\beta}{2}-\frac{1}{2})} + \frac{(2\rho+\alpha-\beta)\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})}{\Gamma(\rho-\frac{\beta}{2})\Gamma(\rho-\frac{\alpha}{2}-\frac{1}{2})}\right] \quad (2.2)$$

Where $Re(\rho) > 0, Re(2\rho-\alpha-\beta) > 0$, a and b are constants , such the expression

$1+ax+b(1-x)$ is not zero.

$$\int_0^{\pi/2} e^{i(2w+1)\pi\theta}(sin\theta)^{w-1}(cos\theta)^{w-1} {}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]\mathrm{d}\theta$$

$$= \frac{e^{i\pi(w+1)/2}\Gamma(w)\Gamma(w-\frac{\alpha'-\beta'}{2})\Gamma(\frac{\alpha'-\beta'}{2}+1)}{2^{2w-\alpha'-\beta'+2}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}$$

$$\times\left[\frac{(2w-\alpha'-\beta')\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})}{\Gamma(w-\frac{\alpha'}{2}+1)\Gamma(w-\frac{\beta'-1}{2})}-\frac{(2w+\alpha'-\beta')\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})}{\Gamma(w-\frac{\beta'}{2}+1)\Gamma(w-\frac{\alpha'-1}{2})}\right] \tag{2.3}$$

where $Re(w)>0$ and $Re(2w-\alpha'-\beta')>0$

$$\int_0^{\pi/2} e^{i(2w+1)\pi\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'}{2};e^{i\theta}cos\theta\right]\mathrm{d}\theta$$

$$=\frac{e^{i\pi(w+1)/2}\Gamma(w-1)\Gamma(w-\frac{\alpha'-\beta'}{2}-1)\Gamma(\frac{\alpha'+\beta'}{2})}{2^{2w-\alpha'-\beta'}\Gamma(\alpha')\Gamma(\beta')}$$

$$\times\left[\frac{(2w-\alpha'-\beta'-2)\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})}{\Gamma(w-\frac{\alpha'}{2})\Gamma(w-\frac{\beta'+1}{2})}-\frac{(2w+\alpha'-\beta'-2)\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})}{\Gamma(w-\frac{\beta'}{2})\Gamma(w-\frac{\alpha'+1}{2})}\right] \tag{2.4}$$

where $Re(w)>0$ and $Re(2w-\alpha'-\beta')>0$

3) Finite double integrals

We evaluate the following four finite double integrals involving hypergeometric functions and Aleph-function of two variables.

$$1)\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+(1-b)]^{-2\rho-1}{}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$

$$e^{i(2w+1)\pi\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]$$

$$\aleph^{0,n:X}_{p_i,q_i,\tau_i;r:Y}\left(\begin{matrix}\mathrm{z}_1x^{\rho_1}(1-x)^{\rho_1}[1+ax+(1-b)]^{-2\rho_1}e^{2w_1i\theta}(sin\theta)^{w_1}(cos\theta)^{w_1}\\ \mathrm{z}_2x^{\rho_2}(1-x)^{\rho_2}[1+ax+(1-b)]^{-2\rho_2}e^{2w_2i\theta}(sin\theta)^{w_2}(cos\theta)^{w_2}\end{matrix}\ \middle|\ \begin{matrix}\mathrm{U}\\ \mathrm{V}\end{matrix}\right)\mathrm{d}\theta\,\mathrm{d}x$$

$$=\frac{2^{\alpha+\beta-2\rho-2}\Gamma(\frac{\alpha+\beta+2}{2})}{\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\aleph_1-\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})\aleph_2\right]$$

$$\times\frac{e^{i\pi(w+1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w-\alpha'-\beta'}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})\aleph_3-\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})\aleph_4\right] \tag{3.1}$$

Where $\aleph_1,\aleph_2,\aleph_3$ and $\aleph_4$ are given as follow :

$$\aleph_1=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{A}_1\\ \mathrm{A}_2\end{matrix}\ \middle|\ \begin{matrix}(\alpha-\beta-2\rho;2\rho_1,2\rho_2),(1+(\alpha+\beta)/2-\rho;\rho_1,\rho_2),\\ (1-2\rho+\alpha+\beta;2\rho_1,2\rho_2),(\alpha/2-\rho;\rho_1,\rho_2),\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\rho;\rho_1,\rho_2),U\\ ((\beta+1)/2-\rho;\rho_1,\rho_2),V\end{matrix}\right) \tag{3.2}$$

$$\aleph_2=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{A}_1\\ \mathrm{A}_2\end{matrix}\ \middle|\ \begin{matrix}(-\alpha+\beta-2\rho;2\rho_1,2\rho_2),(1+(\alpha+\beta)/2-\rho;\rho_1,\rho_2),\\ (1-2\rho-\alpha+\beta;2\rho_1,2\rho_2),((\alpha+1)/2-\rho;\rho_1,\rho_2),\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\rho;\rho_1,\rho_2),U\\(\beta/2-\rho;\rho_1,\rho_2),V\end{matrix}\right) \tag{3.3}$$

$$\text{with } A_j=\frac{4^{-(\rho_j+1)}}{(1+a)^{\rho_j}(1+b)^{\rho_j}}\qquad j=1,2 \tag{3.4}$$

$$\aleph_3=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{B}_1\\\mathrm{B}_2\end{matrix}\middle|\begin{matrix}(\alpha'-\beta'-2w;2w_1,2w_2),(1+(\alpha'+\beta')/2-w;w_1,w_2),\\(1-2w+\alpha'+\beta';2w_1,2w_2),(\alpha'/2-w;w_1,w_2),\end{matrix}\right.$$

$$\left.\begin{matrix}(1-w;w_1,w_2),U\\((\beta'+1)/2-w;w_1,w_2),V\end{matrix}\right) \tag{3.5}$$

$$\aleph_4=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{B}_1\\\mathrm{B}_2\end{matrix}\middle|\begin{matrix}(-\alpha'+\beta'-2w;2w_1,2w_2),(1+(\alpha'+\beta')/2-w;w_1,w_2),\\(1-2w-\alpha'+\beta';2w_1,2w_2),((\alpha'+1)/2-w;w_1,w_2),\end{matrix}\right.$$

$$\left.\begin{matrix},(1-w;w_1,w_2),U\\,(\beta'/2-w;w_1,w_2),V\end{matrix}\right) \tag{3.6}$$

$$\text{with } B_j=\frac{z_1e^{i\pi w_j/2}}{4^{w_j}}\qquad j=1,2 \tag{3.7}$$

The validity conditions of 1) are the following :

$$Re(\rho)>0, Re(w)>0,\ |Arg(z_1)|<\frac{\pi}{2}A, |Arg(z_2)|<\frac{\pi}{2}B\,,$$

$$Re(2\rho-\alpha-\beta+2\rho_1\min_{1\leqslant j\leqslant n_1}\frac{c_j}{\gamma_j}+2\rho_2\min_{1\leqslant j\leqslant n_2}\frac{e_j}{E_j})>0 \text{ and}$$

$$Re(2w-\alpha'-\beta'+2w_1\min_{1\leqslant j\leqslant n_1}\frac{c_j}{\gamma_j}+2w_2\min_{1\leqslant j\leqslant n_2}\frac{e_j}{E_j})>0$$

$$2\,)\int_0^1\int_0^{\pi/2}x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho+1}{}_2F_1\Big[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\Big]$$

$$e^{i\pi(2w-1)\theta}(sin\theta)^{w-2}(cos\theta)^{w-1}\,{}_2F_1\Big[\alpha',\beta';\frac{\alpha'+\beta'}{2};e^{i\theta}cos\theta\Big]$$

$$\times\aleph^{0,n:X}_{p_i,q_i,\tau_i;r:Y}\left(\begin{matrix}\mathrm{z}_1x^{\rho_1}(1-x)^{\rho_1}[1+ax+(1-b)]^{-2\rho_1}e^{2w_1i\theta}(sin\theta)^{w_1}(cos\theta)^{w_1}\\\mathrm{z}_2x^{\rho_2}(1-x)^{\rho_2}[1+ax+(1-b)]^{-2\rho_2}e^{2w_2i\theta}(sin\theta)^{w_2}(cos\theta)^{w_2}\end{matrix}\middle|\begin{matrix}\mathrm{U}\\\mathrm{V}\end{matrix}\right)\mathrm{d}\theta\,\mathrm{d}x$$

$$=\frac{2^{\alpha+\beta-2\rho-1}\Gamma(\frac{\alpha+\beta+2}{2})}{\Gamma(\alpha)\Gamma(\beta)(1+a)^{\rho}(1+b)^{\rho}}\Big[\Gamma(\frac{\alpha+\beta}{2})\,\Gamma(\frac{\beta}{2})\,\aleph_5-\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+\beta}{2})\aleph_6\Big]$$

$$\times\frac{e^{i\pi(w+1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w-\alpha'+1}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}\Big[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})\,\aleph_7-\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'+2}{2})\,\aleph_8\Big] \tag{3.8}$$

$$\aleph_5=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{A}_1\\\mathrm{A}_2\end{matrix}\middle|\begin{matrix}(2+\alpha-\beta-2\rho;2\rho_1,2\rho_2),(2+(\alpha+\beta)/2-\rho;\rho_1,\rho_2),\\(3-2\rho+\alpha+\beta;2\rho_1,2\rho_2),((\alpha+2)/2-\rho;\rho_1,\rho_2),\end{matrix}\right.$$

$$\left.\begin{matrix} (2-\rho;\rho_1,\rho_2),U \\ ((\beta+3)/2-\rho;\rho_1,\rho_2),V \end{matrix}\right) \tag{3.9}$$

$$\aleph_6=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{A}_1\\ \mathrm{A}_2\end{matrix}\middle|\begin{matrix}(2-\alpha+\beta-2\rho;2\rho_1,2\rho_2),(2+(\alpha+\beta)/2-\rho;\rho_1,\rho_2),\\ (3-2\rho-\alpha+\beta;2\rho_1,2\rho_2),((\alpha+3)/2-\rho;\rho_1,\rho_2),\end{matrix}\right.$$

$$\left.\begin{matrix} (2-\rho;\rho_1,\rho_2),U \\ ((\beta+2)/2-\rho;\rho_1,\rho_2),V \end{matrix}\right) \tag{3.10}$$

A_j are given in (3.4)

$$\aleph_7=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{B}_1\\ \mathrm{B}_2\end{matrix}\middle|\begin{matrix}(2+\alpha'-\beta'-2w;2w_1,2w_2),(2+(\alpha'+\beta')/2-w;w_1,w_2),\\ (3-2w+\alpha'+\beta';2w_1,2w_2),(1+\alpha'/2-w;w_1,w_2),\end{matrix}\right.$$

$$\left.\begin{matrix} ,(2-w;w_1,w_2),U \\ ,((\beta'+3)/2-w;w_1,w_2),V \end{matrix}\right) \tag{3.11}$$

$$\aleph_8=\aleph^{0,n+3:X}_{p_i+3,q_i+3,\tau_i;r:Y}\left(\begin{matrix}\mathrm{B}_1\\ \mathrm{B}_2\end{matrix}\middle|\begin{matrix}(2-\alpha'+\beta'-2w;2w_1,2w_2),(2+(\alpha'+\beta')/2-w;w_1,w_2),\\ (3-2w-\alpha'+\beta';2w_1,2w_2),((3+\alpha')/2-w;w_1,w_2),\end{matrix}\right.$$

$$\left.\begin{matrix} ,(2-w;w_1,w_2),U \\ ,((\beta'+2)/2-w;w_1,w_2),V \end{matrix}\right) \tag{3.11}$$

B_j are given in (3.7)

The validity conditions of 2) are the following :

$$Re(\rho)>0, Re(w)>1\,,\ |Arg(z_1)|<\frac{\pi}{2}A, |Arg(z_2)|<\frac{\pi}{2}B\,,$$

$$Re(2\rho-\alpha-\beta+2\rho_1\min_{1\leqslant j\leqslant n_1}\frac{c_j}{\gamma_j}+2\rho_2\min_{1\leqslant j\leqslant n_2}\frac{e_j}{E_j})>2 \text{ and}$$

$$Re(2w-\alpha'-\beta'+2w_1\min_{1\leqslant j\leqslant n_1}\frac{c_j}{\gamma_j}+2w_2\min_{1\leqslant j\leqslant n_2}\frac{e_j}{E_j})>2$$

3) $$\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho+1}{}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$

$$\times e^{i\pi(2w-1)\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'}{2};e^{i\theta}cos\theta\right]$$

$$\times\aleph^{0,n:X}_{p_i,q_i,\tau_i;r:Y}\left(\begin{matrix}\mathrm{z}_1x^{\rho_1}(1-x)^{\rho_1}[1+ax+(1-b)]^{-2\rho_1}e^{2w_1i\theta}(sin\theta)^{w_1}(cos\theta)^{w_1}\\ \mathrm{z}_2x^{\rho_2}(1-x)^{\rho_2}[1+ax+(1-b)]^{-2\rho_2}e^{2w_2i\theta}(sin\theta)^{w_2}(cos\theta)^{w_2}\end{matrix}\middle|\begin{matrix}\mathrm{U}\\ \mathrm{V}\end{matrix}\right)\mathrm{d}\theta\,\mathrm{d}x$$

$$=\frac{2^{\alpha+\beta-2\rho-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\aleph_1-\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})\aleph_2\right]$$

$$\times\frac{e^{i\pi(w-1)/2}\Gamma(\frac{\alpha'+\beta'}{2})}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\aleph_7-\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})\aleph_8\right] \tag{3.12}$$

where $\aleph_1, \aleph_2, \aleph_7$ and $\aleph_8$ are mentioned in (3.2) , (3.3) ,(3.11) and (3.12) respectively an the validity conditions are the following :

$$Re(\rho) > 0, Re(w) > 1, |argz| < \frac{1}{2}\pi\Omega \,,$$

$$Re(2\rho - \alpha - \beta + 2\rho_1 \min_{1\leqslant j\leqslant n_1} \frac{c_j}{\gamma_j} + 2\rho_2 \min_{1\leqslant j\leqslant n_2} \frac{e_j}{E_j}) > 0 \text{ and}$$

$$Re(2w - \alpha' - \beta' + 2w_1 \min_{1\leqslant j\leqslant n_1} \frac{c_j}{\gamma_j} + 2w_2 \min_{1\leqslant j\leqslant n_2} \frac{e_j}{E_j}) > 0$$

$$4\,) \int_0^1 \int_0^{\pi/2} x^{\rho}(1-x)^{\rho-2}\ [1+ax+b(1-x)]^{-2\rho+1} {}_2F_1\Big[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\Big]$$

$$\times e^{i\pi(2w+1)\theta}(sin\theta)^{w}(cos\theta)^{w-1}\ {}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]$$

$$\times \aleph^{0,n:X}_{p_i,q_i,\tau_i;r:Y}\left(\begin{matrix} z_1 x^{\rho_1}(1-x)^{\rho_1}[1+ax+(1-b)]^{-2\rho_1}e^{2w_1 i\theta}(sin\theta)^{w_1}(cos\theta)^{w_1} \\ z_2 x^{\rho_2}(1-x)^{\rho_2}[1+ax+(1-b)]^{-2\rho_2}e^{2w_2 i\theta}(sin\theta)^{w_2}(cos\theta)^{w_2} \end{matrix}\ \middle|\ \begin{matrix} \mathrm{U} \\ \mathrm{V} \end{matrix}\right) \mathrm{d}\theta\, \mathrm{d}x$$

$$= \frac{2^{\alpha+\beta-2\rho-1}\Gamma(\frac{\alpha+\beta}{2})\Gamma(\frac{\alpha+\beta+2}{2})}{\Gamma(\alpha)\Gamma(\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\aleph_5 - \Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})\aleph_6\right]$$

$$\times \frac{e^{i\pi(w+1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')\Gamma(\alpha'-\beta')}\left[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})\,\aleph_3 - \Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})\aleph_4\right] \tag{3.13}$$

Where$\aleph_5, \aleph_6, \aleph_3$ and $\aleph_4$ are mentioned (3.5) , (3.6) , (3.8) and (3.9) respectively an the validity conditions are the following :

$$Re(\rho) > 0, Re(w) > 1, |argz| < \frac{1}{2}\pi\Omega \,,$$

$$Re(2\rho - \alpha - \beta + 2\rho_1 \min_{1\leqslant j\leqslant n_1} \frac{c_j}{\gamma_j} + 2\rho_2 \min_{1\leqslant j\leqslant n_2} \frac{e_j}{E_j}) > 0 \text{ and}$$

$$Re(2w - \alpha' - \beta' + 2w_1 \min_{1\leqslant j\leqslant n_1} \frac{c_j}{\gamma_j} + 2w_2 \min_{1\leqslant j\leqslant n_2} \frac{e_j}{E_j}) > 2$$

Proof : To establish (3.1) the Aleph_function of two variables on the left hande side using (1.1) in Mellin-Barnes contour integral and interchanging the order of integration which is justifiable due to absolute convergence of the integrals , we have :

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s_1,s_2)\prod_{j=1}^{2}\phi_j(s_j)\Bigg(\Bigg(\int_0^1 x^{\rho+\rho_1 s_1+\rho_2 s_2-1}(1-x)^{\rho+\rho_1 s_1+\rho_2 s_2}$$

$$[1+ax+(1-b)]^{-(2\rho+2\rho_1 s_1+2\rho_2 s_2+1)}\ {}_2F_1\Big[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\Big]\mathrm{d}x\Bigg)$$

$$\int_0^{\pi/2} e^{i(2w+2w_1 s_1+2w_2 s_2+1)}(sin\theta)^{w+w_1 s_1+w_2 s_2}(cox\theta)^{w+w_1 s_1+w_2 s_2}$$

$$\times\ {}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]\mathrm{d}\theta\Bigg)\mathrm{d}s_1\mathrm{d}s_2$$

We evaluate the inner integrals with the help of (2.1) and (2.3) and applying (1.1) , we get the R.H.S of (3.1) in terms

of product of Aleph-functions of two variables. The other integrals calculate in the similar method.

4) Particular cases

If $a = b$ and , $\tau_i = 1$ for $i = 1$ to r ; $\tau_{i'} = 1$ to r' ; $\tau_{i''} = 1$ to r'' , the Aleph-functions degenere into I-function of two variables defined by Sharma et al [2] , we obtain :

$$\int_0^1 \int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}(1+b)^{-2\rho+1} {}_2F_1\Big[\alpha,\beta;\frac{\alpha+\beta+2}{2};x\Big] e^{i\pi(2w+1)\theta}\,(sin\theta)^{w-2}$$

$$\times(cos\theta)^{w-1} {}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]$$

$$I^{0,n:X}_{p_i,q_i;r:Y}\left(\begin{matrix} z_1 x^{\rho_1}(1-x)^{\rho_1}[1+b)]^{-2\rho_1}e^{2w_1 i\theta}(sin\theta)^{w_1}(cos\theta)^{w_1} \\ z_2 x^{\rho_2}(1-x)^{\rho_2}[1+b)]^{-2\rho_2}e^{2w_2 i\theta}(sin\theta)^{w_2}(cos\theta)^{w_2}\end{matrix}\ \middle|\ \begin{matrix}U\\V\end{matrix}\right)d\theta\,dx$$

$$=\frac{2^{\alpha+\beta-2\rho_1-1}\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(\alpha)\Gamma(\beta)(1+b)^{2\rho}}\left[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\,I_5 - \Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})I_6\right]$$

$$\times\frac{e^{i\pi(w_1-1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})I_7 - \Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'+2}{2})\,I_8\right]$$

Where $I_k = \aleph_k$ for $k = 5,6,7,8$ and $\tau_i = 1$ for $i = 1$ to r ; $\tau_{i'} = 1$ to r' ; $\tau_{i''} = 1$ to r''

Remark : If r = r' = r" = 1 the I_function degenere into the fox's H-function , see Ronghe [1].

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