Fractional derivative formulae involving the generalized Lauricella function, the generalized polynomials and the multivariable Aleph-function II

$F.Y.\ AYANT^1$

1 Teacher in High School , France

ABSTRACT
In this document, we derive two general fractional derivative formulae involving the generalized Lauricella function, the general polynomials and the multivariable Aleph-function have been derivative by using the concept of fractional derivatives in the theory of hypergeometric function

KEYWORDS : Aleph-function of several variables, generalized Lauricella function, contour integral, general polynomial, fractional derivative

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1. Introduction and preliminaries.

.The function multivariable Aleph-function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [6] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{\mathfrak{n}+1,p_i}]:$

$\cdots\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_j^{(1)},\cdots,\beta_j^{(r)})_{m+1,q_i}]:$

$$\left.\begin{array}{l} [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\phi_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\phi_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0 , \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1} \ldots |z_r|^{\alpha_r}), max(|z_1| \ldots |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1} \ldots |z_r|^{\beta_r}), min(|z_1| \ldots |z_r|) \rightarrow \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this document :

$$V = m_1, n_1; \cdots ; m_r, n_r \tag{1.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots , p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots ,\{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_r) = \aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{D} \end{array} \right) \tag{1.12}$$

In the present paper, we will use the following results :

Let $\mathrm{F}\begin{pmatrix} \mathrm{x}_1 \\ \cdots \\ \mathrm{x}_r \end{pmatrix}$ denote the generalized Lauricella function of several complex variables, see Srivastava et al [10]. We have

$$\mathrm{F}\begin{pmatrix} \mathrm{x}_1 \\ \cdots \\ \mathrm{x}_r \end{pmatrix} = \sum_{m_1,\cdots,m_r=0}^{\infty} A(m_1, \cdots , m_r) \frac{x_1^{m_1} \cdots x_r^{m_r}}{m_1! \cdots m_r!} \tag{1.13}$$

Where : $$A(m_1, \cdots , m_r) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j' + \cdots + m_r\theta_j^{(r)}} \prod_{j=1}^{B'}(b_j')_{m_1\phi_j'} \cdots \prod_{j=1}^{B^{(n)}}(b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{m_1\epsilon_j' + \cdots + m_r\epsilon_j^{(r)}} \prod_{j=1}^{D'}(d_j')_{m_1\delta_j'} \cdots \prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}} \tag{1.14}$$

The generalized polynomials defined by Srivastava [8], is given in the following manner :

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}[y_1, \cdots , y_r] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_r)_{M_rK_r}}{K_r!}$$

$$A[N_1, K_1; \cdots ; N_r, K_r] y_1^{K_1} \cdots y_r^{K_r} \tag{1.15}$$

Where $M_1, \cdots , M_r$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots ; N_r, K_r]$ are arbitrary constants, real or complex.

The fractional derivative of a function f (x) of a complex order μ is defined by Oldham et al ([5], (1974 , page 49) in the followin manner :

For simplicity , the special ense of the fractional derivative operator ${}_aD_x^{\mu}$ when $a=0$, will be written D_x^{μ}

Also we have :

$$D_x^{\mu}\left(x^{\lambda}\right)=\frac{d^{\mu}}{dx^{\mu}}\left(x^{\lambda}\right)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}x^{\lambda-\mu} \quad , Re(\lambda)>-1 \tag{1.16}$$

and the binomial expansion

$$(x+\mu)^{\lambda}=\mu^{\lambda}\sum_{m=0}^{\infty}\binom{\lambda}{m}\left(\frac{x}{\mu}\right)^{m} \quad , \quad \left|\frac{x}{\mu}\right|<1 \tag{1.17}$$

In your investigation, we shall use the following result which may be verified from (1.16), binomial theorem and exponential theorem.

$$D_x^{\lambda-\mu}[x^{\lambda+m-1}\prod_{j=1}^{r}(1-a_jx)^{-\alpha_j}]=x^{\mu+m-1}\frac{\Gamma(\lambda+m)}{\Gamma(\mu+m)}F_D^{(r)}[\lambda,\alpha_1,\cdots,\alpha_r;\mu;a_1x,\cdots,a_rx] \tag{1.18}$$

Where $Re(\lambda)>0$, $\max\{|a_1x|.\cdots.|a_rx|\}<1$, and $F_D^{(r)}$ denotes the Lauricella's hypergeometric function of r-variables which is the generalization of Appell's function F_4 of two variables defined by Lauricella [4].

$$D_x^{\lambda-\mu}[x^{\lambda+m-1}exp(a_rx)\prod_{j=1}^{r}(1-a_jx)^{-\alpha_j}]=x^{\mu+m-1}\frac{\Gamma(\lambda+m)}{\Gamma(\mu+m)}$$

$$\times\,\phi_D^{(r)}[\lambda,\alpha_1,\cdots,\alpha_{r-1};\mu;a_1x,\cdots,a_rx] \tag{1.19}$$

Where $Re(\lambda)>0$, $\max\{|a_1x|.\cdots.|a_{r-1}x|\}<1$, and $\phi_D^{(r)}$ denotes the confluent form of Lauricella's hypergeometric function $F_D^{(r)}$.

2. The fractional derivative formula

In the present paper, we use the following notations.

$$A_1=\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_r)_{M_rK_r}}{K_r!}A[N_1,K_1;\cdots;N_r,K_r]$$

$$A_r=\sum_{i=1}^{r}\lambda_i\zeta;B_r=\sum_{i=1}^{r}\mu_i\zeta \quad ; \quad g(x_i)=x_i(1-a_ix)^{-\rho_i},i=1,\cdots,r$$

First formula :

$$D_x^{(\lambda+A_r)-(\mu+B_r)}\left[x^{\lambda-1}(x+\zeta)^{\sigma}\mathrm{F}\begin{pmatrix}\mathrm{g}\ (\mathrm{x}_1)\\ \cdots\\ \mathrm{g}\ (\mathrm{x}_r)\end{pmatrix}S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix}\mathrm{y}_1x^{\sigma_1}(x+\zeta)^{\sigma_1}\\ \cdots\\ \mathrm{y}_rx^{\sigma_r}(x+\zeta)^{\sigma_r}\end{pmatrix}\aleph\begin{pmatrix}\mathrm{z}_1x^{\mu_1}(x+\zeta)^{\lambda_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{\mu_r}(x+\zeta)^{\lambda_r}\end{pmatrix}\right]$$

$$=x^{\mu-1}\zeta^{\sigma}\sum_{m=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{\infty}\sum_{l_1,\cdots,l_r=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}A(k_1,\cdots,k_r)A_1\prod_{i=1}^{r}\frac{x_i^{k_i}(x/\zeta)^my_i^{K_i}}{k_i!l_i!m!}$$

$$\prod_{i=1}^{r}(\rho_i k_i)_{l_i}(a_i x)^{l_i} \aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1 x^{\mu_1}\zeta^{\lambda_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\mu_r}\zeta^{\lambda_r} \end{array}\right|$$

$$\left.\begin{array}{c} (\text{ 1 - }\lambda - m - \sum_{i=1}^{r}(\sigma_i K_i + l_i); \lambda_1, \cdots, \lambda_r),\ (\text{ -}\sigma - \sum_{i=1}^{r}\sigma_i K_i; \lambda_1, \cdots, \lambda_r), A : C \\ \cdots \qquad\qquad\qquad \cdots \\ (\text{1 - }\mu - m - \sum_{i=1}^{r}(\sigma_i K_i + l_i); \mu_1, \cdots, \mu_r), (\text{ m-}\sigma - \sum_{i=1}^{r}\sigma_i K_i; \lambda_1, \cdots, \lambda_r), B : D \end{array}\right) \tag{2.1}$$

The validity conditions are the following :

a) $Re(\lambda_i) > 0,\ |arg(x/\zeta)| < \pi, i = 1, \cdots, r$

b) $Re[\lambda + \sum_{i=1}^{r}\lambda_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$, $Re[\sigma + \sum_{i=1}^{r}\lambda_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1, j = 1, \cdots, m_k$

c) $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

Second formula

$$D_x^{(\lambda+A_r)-(\mu+B_r)}\left[x^{\lambda-1}(x+\zeta)^{\sigma}e^{a_r x}\mathrm{F}\left(\begin{array}{c}\mathrm{g}\ (\mathrm{x}_1)\\ \cdots \\ \mathrm{g}\ (\mathrm{x}_{r-1})\\ \mathrm{x}_r\end{array}\right)S^{M_1,\cdots,M_r}_{N_1,\cdots,N_r}\left(\begin{array}{c}\mathrm{y}_1 x^{\sigma_1}(x+\zeta)^{\sigma_1}\\ \cdots \\ \mathrm{y}_r x^{\sigma_r}(x+\zeta)^{\sigma_r}\end{array}\right)\aleph\left(\begin{array}{c}\mathrm{z}_1 x^{\mu_1}(x+\zeta)^{\lambda_1}\\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\mu_r}(x+\zeta)^{\lambda_r}\end{array}\right)\right]$$

$$= x^{\mu-1}\zeta^{\sigma}\sum_{m=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{\infty}\sum_{l_1,\cdots,l_r=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}A(k_1,\cdots,k_r)A_1\prod_{i=1}^{r}\frac{(x/\zeta)^m x_i^{k_i}y_i^{K_i}}{k_i!m!}$$

$$\prod_{i=1}^{r-1}\frac{(\rho_i k_i)_{l_i}(a_j x)^{l_i}}{l_i!}\frac{(a_r x)^{l_r}}{l_r!}\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1 x^{\mu_1}\zeta^{\lambda_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\mu_r}\zeta^{\lambda_r} \end{array}\right|$$

$$\left.\begin{array}{c} (\text{ 1 - }\lambda - m - \sum_{i=1}^{r}(\sigma_i K_i + l_i); \lambda_1, \cdots, \lambda_r),\ (\text{ -}\sigma - \sum_{i=1}^{r}\sigma_i K_i; \lambda_1, \cdots, \lambda_r), A : C \\ \cdots \qquad\qquad\qquad \cdots \\ (\text{1 - }\mu - m - \sum_{i=1}^{r}(\sigma_i K_i + l_i); \mu_1, \cdots, \mu_r), (\text{ m-}\sigma - \sum_{i=1}^{r}\sigma_i K_i; \lambda_1, \cdots, \lambda_r), B : D \end{array}\right) \tag{2.2}$$

The validity conditions are the following :

a) $Re(\lambda_i) > 0,\ |arg(x/\zeta)| < \pi, i = 1, \cdots, r$

b) $Re[\lambda + \sum_{i=1}^{r}\lambda_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$, $Re[\sigma + \sum_{i=1}^{r}\lambda_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1, j = 1, \cdots, m_k$

c) $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

Proof:

To prove (2.1), we first applying the definition of the generalized Lauricella function, the generalized polynomials and multivariable Aleph-function by its Mellin-barnes contour integral occuring on the left-hand sides, collect the power of x and $x+\zeta$ and applying the binomial expansion (1.17). Changing the order of summations, integrations and fractional derivative operators and making use the formula (1.16) and again if apply the definition (1.1), we shall then get the desired formula (2.1).

Similarly (2.2) can be provided by using the definition of exponential serie.

3. Paticulars cases

If $\tau_i = \tau_{i^{(k)}} = 1$, the Aleph-function of several variables degenere in the I-function of several variables defined by Sharma and Ahmad [6]. For more details, see Tiwari [11].

If $\tau_i = \tau_{i^{(k)}} = 1$ and $R = R^{(1)} =, \cdots, R^{(r)} = 1$, the multivariable Aleph-function degenere in multivariable H-function, see Srivastava et al [9]. We obtain

First formula

$$D_x^{(\lambda+A_r)-(\mu+B_r)}\left[x^{\lambda-1}(x+\zeta)^{\sigma}\mathrm{F}\begin{pmatrix}\mathrm{g}\ (\mathrm{x}_1)\\ \cdots\\ \mathrm{g}\ (\mathrm{x}_r)\end{pmatrix}S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix}\mathrm{y}_1x^{\sigma_1}(x+\zeta)^{\sigma_1}\\ \cdots\\ \mathrm{y}_rx^{\sigma_r}(x+\zeta)^{\sigma_r}\end{pmatrix}H\begin{pmatrix}\mathrm{z}_1x^{\mu_1}(x+\zeta)^{\lambda_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{\mu_r}(x+\zeta)^{\lambda_r}\end{pmatrix}\right]$$

$$=x^{\mu-1}\zeta^{\sigma}\sum_{m=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{\infty}\sum_{l_1,\cdots,l_r=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}A(k_1,\cdots,k_r)A_1\prod_{i=1}^{r}\frac{x_i^{k_i}(x/\zeta)^m y_i^{K_i}}{k_i!l_i!m!}$$

$$\prod_{i=1}^{r}(\rho_ik_i)_{l_i}(a_ix)^{l_i}\ H_{p+2,q+2:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix}\mathrm{z}_1x^{\mu_1}\zeta^{\lambda_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{\mu_r}\zeta^{\lambda_r}\end{matrix}\ \middle|\right.$$

$$\left.\begin{matrix}(\ 1\text{ - }\lambda-m-\sum_{i=1}^{r}(\sigma_iK_i+l_i);\lambda_1,\cdots,\lambda_r),\ (\ \text{-}\sigma-\sum_{i=1}^{r}\sigma_iK_i;\lambda_1,\cdots,\lambda_r),A':C'\\ \cdots\\ (1\text{ - }\mu-m-\sum_{i=1}^{r}(\sigma_iK_i+l_i);\mu_1,\cdots,\mu_r),(\ \text{m-}\sigma-\sum_{i=1}^{r}\sigma_iK_i;\lambda_1,\cdots,\lambda_r),B':D'\end{matrix}\right) \tag{3.1}$$

The validity conditions are the following :

a) $Re(\lambda_i) > 0,\ |arg(x/\zeta)| < \pi, i = 1,\cdots,r$

b) $Re[\lambda+\sum_{i=1}^{r}\lambda_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$, $Re[\sigma+\sum_{i=1}^{r}\lambda_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1, j = 1,\cdots,m_k$

c) $|argz_i| < \frac{1}{2}A_i\pi$, where : $A_i = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(i)} - \sum_{j=\mathfrak{n}+1}^{p}\alpha_j^{(i)} - \sum_{j=1}^{q}\beta_j^{(i)} + \sum_{j=1}^{n_i}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}$

$$+\sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}\delta_j^{(i)} > 0 \text{, with } i=1,\cdots,r$$

Second formula

$$D_x^{(\lambda+A_r)-(\mu+B_r)}\left[x^{\lambda-1}(x+\zeta)^{\sigma}e^{a_r x}\mathrm{F}\begin{pmatrix}\mathrm{g}\ (\mathrm{x}_1)\\ \cdots\\ \mathrm{g}\ (\mathrm{x}_{r-1})\\ \mathrm{x}_r\end{pmatrix}S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix}\mathrm{y}_1 x^{\sigma_1}(x+\zeta)^{\sigma_1}\\ \cdots\\ \mathrm{y}_r x^{\sigma_r}(x+\zeta)^{\sigma_r}\end{pmatrix}H\begin{pmatrix}\mathrm{z}_1 x^{\mu_1}(x+\zeta)^{\lambda_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r x^{\mu_r}(x+\zeta)^{\lambda_r}\end{pmatrix}\right]$$

$$=x^{\mu-1}\zeta^{\sigma}\sum_{m=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{\infty}\sum_{l_1,\cdots,l_r=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}A(k_1,\cdots,k_r)A_1\prod_{i=1}^{r}\frac{(x/\zeta)^m x_i^{k_i}y_i^{K_i}}{k_i!m!}$$

$$\prod_{i=1}^{r-1}\frac{(\rho_i k_i)_{l_i}(a_j x)^{l_i}}{l_i!}\frac{(a_r x)^{l_r}}{l_r!}H_{p+2,q+2:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|}\mathrm{z}_1 x^{\mu_1}\zeta^{\lambda_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r x^{\mu_r}\zeta^{\lambda_r}\end{array}\right.$$

$$\left.\begin{array}{c}(1-\lambda-m-\sum_{i=1}^{r}(\sigma_i K_i+l_i);\lambda_1,\cdots,\lambda_r),\ (-\sigma-\sum_{i=1}^{r}\sigma_i K_i;\lambda_1,\cdots,\lambda_r),A':C'\\ \cdots \qquad \cdots\\ (1-\mu-m-\sum_{i=1}^{r}(\sigma_i K_i+l_i);\mu_1,\cdots,\mu_r),(\mathrm{m}-\sigma-\sum_{i=1}^{r}\sigma_i K_i;\lambda_1,\cdots,\lambda_r),B':D'\end{array}\right) \qquad (3.2)$$

The validity conditions are the following :

a) $Re(\lambda_i)>0,\ |arg(x/\zeta)|<\pi, i=1,\cdots,r$

b) $Re[\lambda+\sum_{i=1}^{r}\lambda_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>-1$, $Re[\sigma+\sum_{i=1}^{r}\lambda_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>-1, j=1,\cdots,m_k$

c) $|argz_i|<\frac{1}{2}A_i\pi$, where : $A_i=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(i)}-\sum_{j=\mathfrak{n}+1}^{p}\alpha_j^{(i)}-\sum_{j=1}^{q}\beta_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}$

$$+\sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}\delta_j^{(i)} > 0 \text{, with } i=1,\cdots,r$$

4. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as I-function of several variables defined by Sharma and Ahmad [6] , multivariable H-function , see Srivastava et al [9] , the Aleph-function of two variables defined by K.sharma [7], the I-function of two variables defined by Goyal and Agrawal [1,2,3] , ,and the h-function of two variables , see Srivastava et al [9].

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Personal adress : 411 Avenue Joseph Raynaud

Le parc Fleuri , Bat B

83140 , Six-Fours les plages

Tel : 06-83-12-49-68

Department : VAR

Country : FRANCE