NEW GENERALIZATION OF FRACTIONAL KINETIC EQUATION USING ALEPH-FUNCTION OF TWO VARIABLES

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Abstract : Recently, Dutta et al [21] use the Aleph-function on one variable for solving generalized fractional kinetic equation. In this paper, the solution of a class of fractional Kinetic equation involving Aleph-function of two variables has been discussed. Special cases involving the I-function of two variables , H-function of two variables and product of two Aleph functions are also discussed. Results obtained are related to recent investigations of possible astrophysical solutions of the solar neutrino problem.

Keywords : Aleph function of two variables. I-function of two variables. H-function of two variables. Aleph-function of one variables. Fractional Kinetic equation. Laplace transform. Riemann-Liouville fractional integral.

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1. INTRODUCTION

The fractional calculus has many important developments and concepts in mathematics initiates with fractional kinetic models (kinetic equation). The great use of Mathematical Physics in imposing astrophysical problems have pulled stargazers and physicists to pay more attention in available mathematical tools that can be used to solve several problems of astrophysics. The importance of fractional kinetic equation has been increased by virtue of its occurrence in certain problem related to kinetic motion of particles in science and engineering. The thermal and hydrostatics equilibrium are pretended as spherically symmetric, non-magnetic, non-rotating, self gravitating model of a star like sun. The properties of star arecharacterized by its mass, brightness effective surface temperature, radius, central density, temperature etc. Turn over an arbitrary reaction characterized by N = N (t) which is dependent on time. It is possible to compute rate of change dN/dt to a balance between the demolition rate d and the production rate p of N , that is
dN/dt = −d + p. In general, through interaction mechanism, demolition and production depend on the quantity N itself :
d = d (N) or p = p (N). This dependence is complicated for the demolition of production at time depends not only on N(t), but also on the proceding history N (ı̄) , ı̄ < t , of the variable N.
This may be formally represented by [3].

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{1.1}$$

Where N_t denote the function defined by N_t (ı̇) = N (t – ı̇) , ı̇ > 0.
Haubold and Mathai [3] studied a special case of this equation , when instance of changes in quantity N (t) are unvalued , is given by the equation :

$$\frac{dN_i}{dt} = -c_i N_i(t) \tag{1.2}$$

with the initial condition that $N_t(t=0) = N_0$ is the number density of species i at time t = 0; constant $c_i > 0$, known as standard kinetic equation. The solution of the Eq. (2) is give :

$$N_i(t) = N_0 exp(-c_i t)$$

Alternative form of Eq. (2) can be obtained on integration :

$$N(t) - N_0 = c_0 D_t^{-1} N(t) \tag{1.3}$$

where ${}_0D_t^{-1}$ is the standard integral operator. Haubold and Mathai [3] have given the fractional generalization of the standard kinetic Eq, (2) as

$$N(t) - N_0 = c_0 D_t^{-\upsilon} N(t) \tag{1.4}$$

Where ${}_0D_t^{-\upsilon}$ is the well known Riemann-Liouville fractional integral operator (Oldhman and Spanier [3] ; Samko et al [4] ; Miller and Ross [19] , Srivastava and Saxena [18]) defined by :

$${}_0D_t^{-\upsilon} = \frac{1}{\Gamma(\upsilon)} \int_0^t (t-u)^{\upsilon-1} f(u)\,\mathrm{d}u\,, \qquad \mathrm{Re}\,(\,\upsilon\,) > 0 \tag{1.5}$$

The solution of the fractional Kinetic Eq. (1.4) is given by Haubold and Mathai [3] as:

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\upsilon n+1)} (ct)^{n\upsilon} \tag{1.6}$$

Further, a number of research workers have also studied the generalization of kinetic equation in term of Mittag-Leffler functions. Recently, Chaurasia and Kumar [20] investigated the solution of the fractional kinetic equation associated with the generalized M-series.

2. Mathematical Prerequisites

Recently, I -function of two variables has been introduced and studied by Sharma et al.[15], which is a generalization of the H-function of two variables due to Gupta et al.[4] , has been investigated the certain double integrals involving H-function of two variables due to Buschman et al. These double integrals are of most general character and can be suitably specialized to yield a number of known or new integral formulae of much interest to mathematical analysis which are likely to prove quite useful to solve so me typical boundary value problems. The double Barnes integral occurring in the paper will be referred to as the Aleph-function of two variables throughout our present study and will be defined and represented as follows , see K. Sharma [16] :

$$\aleph[x,y] = \aleph^{0,n:m_1,n_1:m_2,n_2}_{P_i,Q_i,\tau_i:r;P_i',Q_i',\tau_i':r';P_i",Q_i",\tau_i":r"}[z_1,z_2]$$

$$= \aleph^{0,n:m_1,n_1:m_2,n_2}_{P_i,Q_i,\tau_i:r;P_i',Q_i',\tau_i':r';P_i",Q_i",\tau_i":r"}\left(z_1,z_2 \,\middle|\, \begin{matrix} \mathrm{A}(\tau_i):C(\tau_{i'});E(\tau_{i''}) \\ \mathrm{B}(\tau_i):D(\tau_{i'});F(\tau_{i''}) \end{matrix}\right)$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)\, z_1^{s_1} z_2^{s_2}\, \mathrm{d}s_1 \mathrm{d}s_2 \tag{2.1}$$

where

$$A(\tau_i) = (a_j,\alpha_j,A_j)_{1,n}, [\tau_j(a_{ji},\alpha_{ji},A_{ji})]_{n+1,P_i}\ ; B(\tau_i) = [\tau_j(b_{ji},\beta_{ji},B_{ji})]_{1,Q_i} \qquad (2,2)$$

$$C(\tau_{i'}) = (c_j,\gamma_j)_{1,n_1}, [\tau_j(c_{ji'},\gamma_{ji'})]_{n_1+1,P_{i'}}; D(\tau_{i'}) = (d_j,\delta_j)_{1,m_1}, [\tau_j(d_{ji'},\delta_{ji'})]_{m_1+1,P_{i'}} \qquad (2,3)$$

$$E(\tau_{i''}) = (e_j,E_j)_{1,n_2}, [\tau_j(e_{ji''},E_{ji''})]_{n_2+1,P_{i''}}; F(\tau_{i''}) = (f_j,F_j)_{1,m_2}, [\tau_j(f_{ji''},F_{ji''})]_{m_2+1,Q_{i''}} \qquad (2.4)$$

In the sequel we will use this notation. The defined integral of the above function, the conditions concerning the parameters the existence and convergence conditions, see K.Sharma [16]. Throughout the present document, we assume that the existence and convergence conditions of the Aleph-function of two variables.

3. GENERALIZED FRACTIONAL KINETIC EQUATION

Lemma 3.1 The Laplace transform of the Aleph-function of two variables as follows

$$L\{\, t^{\lambda-1}\, \aleph^{0,n:m_1,n_1:m_2,n_2}_{P_i,Q_i,\tau_i:r;P'_i,Q'_i,\tau'_i:r';P_i",Q_i",\tau_i":r"}\,(z_1t^{\upsilon}, z_2t^{\upsilon}\,)\} = u^{-\lambda}\times\cdots$$

$$\aleph^{0,n+1:m_1,n_1:m_2,n_2}_{P_i+1,Q_i,\tau_i:r;P'_i,Q'_i,\tau'_i:r';P_i",Q_i",\tau_i":r"}\left(\frac{z_1}{u^{\upsilon}}, \frac{z_2}{u^{\upsilon}}\,\middle|\, \begin{matrix} (1\text{-}\lambda;\upsilon,\upsilon), A(\tau_i) : C(\tau_{i'}); E(\tau_{i''}) \\ \mathrm{B}(\tau_i) : D(\tau_{i'}); F(\tau_{i''}) \end{matrix}\right) \tag{3.1}$$

Where u , $z_1, z_2, \lambda, \upsilon \in \mathbb{C}, Re(u) > 0 \quad \tau_i > 0 \quad i\in\overline{[1,r]},\ \tau'_i > 0, i'\in\overline{[1,r']}$, $\tau_i" > 0, i"\in\overline{[1,r"]}$

Proof. For convenience , we denote the left side of (3.1) by Ł.

$$Ł = \frac{1}{(2\pi\omega)^2}\int_0^{\infty} exp(-ut)t^{\lambda-1}\int_{L_1}\int_{L_2} t^{\upsilon(s_1+s_2)}\, z_1^{s_1} z_2^{s_2}\phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t$$

Changing the order of integration, which is permissible under the stated conditions and applied the formula of Laplace Transform , we have :

$$Ł\ = \frac{u^{-\lambda}}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} z_1^{s_1} z_2^{s_2}\,\phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)\,u^{-\upsilon(s_1+s_2)}\,\Gamma(\lambda+\upsilon(s_1+s_2))\mathrm{d}s_1\mathrm{d}s_2$$

After simple adjustement we finally arrived at (3.1).

Lemma 3.2 From the Lemma 3.1 it is clear that :

$$L^{-1}\,\{\, x^{-\lambda}\,\aleph^{0,n:m_1,n_1:m_2,n_2}_{P_i,Q_i,\tau_i:r;P'_i,Q'_i,\tau'_i:r';P_i",Q_i",\tau_i":r"}\,(z_1x^{\upsilon}, z_2x^{\upsilon})\,\} = t^{\lambda-1}\times \tag{3.2}$$

$$\aleph^{0,n:m_1,n_1:m_2,n_2}_{P_i,Q_i+1,\tau_i:r;P'_i,Q'_i,\tau'_i:r';P_i",Q_i",\tau_i":r"}\left(\frac{z_1}{t^{\upsilon}}, \frac{z_2}{t^{\upsilon}}\,\middle|\, \begin{matrix} \mathrm{A}(\tau_i) : C(\tau_{i'}); E(\tau_{i''}) \\ (\lambda;\upsilon,\upsilon), B(\tau_i) : D(\tau_{i'}); F(\tau_{i''}) \end{matrix}\right)$$

Where x,$z_1, z_2, \lambda, \upsilon \in \mathbb{C}\ Re(x) > 0, \upsilon > 0\ \ \tau_i > 0$, $i\in\overline{[1,r]},\ \tau'_i > 0, i'\in\overline{[1,r']}$, $\tau_i" > 0, i"\in\overline{[1,r"]}$

Theorem3,3 If $\upsilon > 0$, c > 0, d > 0 , $\mu > 0$,$Re(s) >|\ d|^{\upsilon/\alpha}$ c # d and $\tau_i > 0$ $i\in\overline{[1,r]},\ \tau'_i > 0, i'\in\overline{[1,r']}$, $\tau_i" > 0, i"\in\overline{[1,r"]}$, then the generalized fractional kinetic equation

$$N(t) - N_0t^{\mu-1}\,\aleph^{0,n:m_1,n_1:m_2,n_2}_{P_i,Q_i,\tau_i:r;P'_i,Q'_i,\tau'_i:r';P_i",Q_i",\tau_i":r"}((dt)^{\upsilon}z_1,(dt)^{\upsilon}z_2) = -c^{\upsilon}{}_0D_t^{-\upsilon}N(t) \tag{3.3}$$

there holds the formula :

$$N(t) = N_0 t^{\mu-1}\sum_{k=0}^{\infty}(-1)^k (ct)^{k\upsilon}\aleph^{0,n_1+1:m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1,\tau_i:r;P_{i'},Q_{i'},\tau_{i'}:r';P_{i"},Q_{i"},\tau_{i"}:r"}\left((dt)^{\upsilon} z_1, (dt)^{\upsilon} z_2 \middle| \begin{matrix} (1\text{-}\mu;\upsilon,\upsilon), A(\tau_i) : C(\tau_{i'}); E(\tau_{i''}) \\ (1\text{-k}\upsilon-\mu;\upsilon,\upsilon), B(\tau_i) : D(\tau_{i'}); F(\tau_{i''}) \end{matrix} \right) \quad (3.4)$$

Proof. Applied Laplace transform of both the sides of Eq. (3.3) and using Lemme 3,1, we get :

$$\mathcal{N}(x) - \frac{N_0}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} z_1^{s_1} z_2^{s_2}\,\phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)\,x^{-\mu-\upsilon(s_1+s_2)}\,d^{\upsilon(s_1+s_2)}$$

$$\Gamma(\mu+\upsilon(s_1+s_2))\,\mathrm{d}s_1\mathrm{d}s_2 = -c^{\mu}x^{-\mu}\,\mathcal{N}(x) \quad (3.5)$$

Solving for $\mathcal{N}(x)$, its gives :

$$\mathcal{N}(x) = \frac{N_0}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} z_1^{s_1} z_2^{s_2}\,\phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)\,x^{-\mu-\upsilon(s_1+s_2)}\,\Gamma(\mu+\upsilon(s_1+s_2))$$

$$d^{\upsilon(s_1+s_2)}\mathrm{d}s_1\mathrm{d}s_2 \times (1+c^{\mu}x^{-\mu})^{-1} \quad (3.6)$$

Now taking inverse Laplace transform both sides of Eq. (3.6) and using Lemme 3,2 , we get the desired result (3.4).

4. Particular Cases

If we put $\tau_i = 1, \tau_i' = 1, \tau_i" = 1 (i \in \overline{[1,r]}$, i' $\in \overline{[1,r']}, i" \in \overline{[1,r"]}$) , then arrive at the following result in the term of I-function of two variables defined by C.K. Sharma and P.L. Mishra [15].

Corollary 4,1 If $\upsilon > 0$, c > 0, d > 0 , $\mu > 0$,$Re(s) >| d|^{\upsilon/\alpha}$ c # d , then the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1}\, I^{0,n:m_1,n_1:m_2,n_2}_{p_i,q_i,r:p_i',q_i',r':p_i",q_i",r"}\left((dt)^{\upsilon} z_1, (dt)^{\upsilon} z_2\right) = -c^{\upsilon}\,{}_0D_t^{-\upsilon}N(t)$$

has the solution of the form :

$$N(t) = N_0 t^{\mu-1}\sum_{k=0}^{\infty}(-1)^k (ct)^{k\upsilon}\, I^{0,n+1:m_1,n_1:m_2,n_2}_{p_i+1,q_i+1,r:p_i',q_i',r':p_i",q_i":r"}\left((dt)^{\upsilon} z_1, (dt)^{\upsilon} z_2 \middle| \begin{matrix} (1\text{-}\mu;\upsilon,\upsilon), A(1) : C(1); E(1) \\ (1\text{-k}\upsilon-\mu;\upsilon,\upsilon), B(1) : D(1); F(1) \end{matrix} \right) \quad (4,2)$$

If you put , $\tau_i = 1, \tau_i' = 1, \tau_i" = 1 (i \in \overline{[1,r]}$, i' $\in \overline{[1,r']}, i" \in \overline{[1,r"]}$ and set $r = r' = r" = 1$, then we arrive at the following result in the term of H-function of two variables : see Gupta and al [4].

Corollary 4,2 If $\upsilon > 0$, c > 0, d > 0 , $\mu > 0$,$Re(s) >| d|^{\upsilon/\alpha}$ c # d , then the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1}\, H^{0,n:m',n':m",n"}_{p,q:p',q':p",q"}\,((dt)^{\upsilon} z_1, (dt)^{\upsilon} z_2) = -c^{\upsilon}\, {}_0D_t^{-\upsilon} N(t) \tag{4.3}$$

has the solution of the form :

$$N(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{k\upsilon}\, H^{0,n+1:m',n':m",n"}_{p+1,q+1:p',q':p",q"} \left(z_1 (dt)^{\upsilon}, z_2 (dt)^{\upsilon} \;\middle|\; \begin{matrix} (1\text{-}\mu; \upsilon, \upsilon), (a_j; \alpha'_j, \alpha''_j)_{1,p}; (c'_j; \gamma'_j)_{1,p_1}; (c''j; \gamma''j)_{1,p_2} \\ (1\text{-k}\upsilon - \mu; \upsilon, \upsilon), (b_j; \beta'_j, \beta''_j)_{1,q} : (d'_j; \delta'_j)_{1,q_1}; (d''_j; \delta''_j)_{1,q_2} \end{matrix} \right) \tag{4.4}$$

If you put $n = p_i = q_i = 0, i \in \overline{[1, r]}$, we obtain the product of two Aleph-functions of one variable. For more details concerning the Aleph-function of one variable , see N. Sudland et al [8] , R.K. Saxena et al [12 , 13] , B.K. Dutta et al [21].

Corollary 4,3 If υ > 0 , c > 0, d > 0 , μ > 0,$Re(s) >| \, d|^{\upsilon/\alpha}$ c # d , then the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1}\, \aleph^{m',n'}_{p'_i,q'_i,\tau'_i:r'}\,((dt)^{\mu} z_1)\, \aleph^{m",n"}_{p_i",q_i",\tau_i":r"}\,((dt)^{\mu} z_2) = -c^{\upsilon}\, {}_0D_t^{-\upsilon} N(t) \tag{4.5}$$

has the solution of the form :

$$N(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{k\upsilon}\, \aleph^{0,1:m',n':m",n"}_{1,1:p'_i,q'_i,\tau'_i:r';p_i",q_i",\tau_i";r"} \left(z_1 (dt)^{\upsilon}, z_2 (dt)^{\upsilon} \;\middle|\; \begin{matrix} (1\text{-}\mu; \upsilon, \upsilon) : C(\tau_{i'}); E(\tau_{i''}) \\ (1\text{-}\mu - k\upsilon; \upsilon, \upsilon) : D(\tau_{i'}); F(\tau_{i''}) \end{matrix} \right) \tag{4.6}$$

5. Conclusion

Aleph-function of two variables is general in nature and includes a number of known and new results as particular cases. This extended fractional kinetic equation can be used to compute the particle reaction rate and may be utilized in other branch of mathematics. Results obtained in this paper provide an extension of [3, 10,11].

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