Summation formulae for multivariable Aleph-function

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ABSTRACT
In this document, we derive a summation formula of a multiple finites series involving multivariable Aleph-function and then discuss some of its paticular cases in the end of this paper.

KEYWORDS : Aleph-function of several variables , summation formula, Mellin-Barnes integral, expansion .

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1. Conclusion

The object of this document is to study a expansions of mulitple finite serie involving the multivariables aleph-function. These function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [6] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} \mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r \end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ ,[\tau_i(b_{ji};\beta_j^{(1)},\cdots,\beta_j^{(r)})_{m+1,q_i}]:$

$\left.\begin{array}{c} [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(u_1,\cdots,u_r)\prod_{k=1}^r\phi_k(u_k)z_k^{u_k}\,\mathrm{d}u_1\cdots\mathrm{d}u_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(u_1,\cdots,u_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}u_k)}{\sum_{i=1}^R[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^r\alpha_{ji}^{(k)}u_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^r\beta_{ji}^{(k)}u_k)]} \tag{1.2}$$

and $$\phi_k(u_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}u_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}u_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}u_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}u_k)]} \tag{1.3}$$

where $j=1$ to r and $k=1$ to r

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R\ , i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} u_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} u_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} u_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg x_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R\ , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_r) = 0(\ |z_1|^{\alpha_1} \ldots |z_r|^{\alpha_r}\ ), max(\ |z_1| \ldots |z_r|\ ) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_r) = 0(\ |z_1|^{\beta_1} \ldots |z_r|^{\beta_r}\ ), min(\ |z_1| \ldots |z_r|\ ) \rightarrow \infty$$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$V = m_1, n_1; \cdots ; m_r, n_r \tag{1.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \vdots & \cdots \\ \mathrm{z}_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.12}$$

2. Main result

$$\sum_{s_1=0}^{k_1} \sum_{t_1=0}^{l_1} \cdots \sum_{s_r=0}^{k_r} \sum_{t_r=0}^{l_r} \frac{(-k_1)_{s_1}(-l_1)_{t_1} \cdots (-k_r)_{s_r}(-l_r)_{t_r}}{s_1!t_1! \cdots s_r!t_r!}$$

$$\frac{(g_1)_{s_1}(h_1)_{t_1} \cdots (g_r)_{s_r}(h_r)_{t_r}}{(g_1 - h_1 - k_1 + 1)_{s_1}(h_1 - g_1 - l_1 + 1)_{t_1} \cdots (g_r - h_r - k_r + 1)_{s_r}(h_r - g_r - l_r + 1)_{t_r}}$$

$$\aleph^{0,\mathfrak{n}+r:V}_{p_i+r,q_i+2r,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & (1 - \mathrm{e}_1 - s_1 - t_1; \alpha'_1, \cdots, \alpha_r^{(r)}), \cdots, (1 - e_r - s_r - t_r; \alpha'_r, \cdots, \alpha_r^{(r)}), \\ \vdots & \cdots \\ \mathrm{z}_r & (1 - \mathrm{e}_1 - s_1; \alpha'_1, \cdots, \alpha_r^{(r)}), \cdots, (1 - e_r - s_r; \alpha'_r, \cdots, \alpha_r^{(r)}), \end{array}\right.$$

$$\left.\begin{array}{c} \mathrm{A} : \mathrm{C} \\ \cdots \\ (1 - \mathrm{e}_1 - t_1; \alpha'_1, \cdots, \alpha_r^{(r)}), \cdots, (1 - e_r - t_r; \alpha'_r, \cdots, \alpha_r^{(r)}), B : D \end{array}\right)$$

$$= \frac{(g_1)_{l_1}(h_1)_{k_1} \cdots (g_r)_{l_r}(h_r)_{k_r}}{(g_1 - h_1)_{l_1}(h_1 - g_1)_{k_1} \cdots (g_r - h_r)_{l_r}(h_r - g_r)_{k_r}}$$

$$\aleph^{0,\mathfrak{n}+r:V}_{p_i+r,q_i+2r,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & (1 - \mathrm{e}_1 - k_1 - l_1; \alpha'_1, \cdots, \alpha_r^{(r)}), \cdots, (1 - e_r - k_r - l_r; \alpha'_r, \cdots, \alpha_r^{(r)}), \\ \vdots & \cdots \\ \mathrm{z}_r & (1 - \mathrm{e}_1 - k_1; \alpha'_1, \cdots, \alpha_r^{(r)}), \cdots, (1 - e_r - k_r; \alpha'_r, \cdots, \alpha_r^{(r)}), \end{array}\right.$$

$$\begin{matrix} \mathrm{A : C} \\ \cdots \\ (1 - \mathrm{e}_1 - l_1; \alpha'_1, \cdots, \alpha_r^{(r)}), \cdots, (1 - e_r - l_r; \alpha'_r, \cdots, \alpha_r^{(r)}), B : D \end{matrix} \Bigg) \quad (2.1)$$

Provided that :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is given in (1.5)

Proof :

We define the multivariable Aleph-function by the Mellin-Barnes integral on the left hand side of the equation (2.1) and then changing the order of integration and summation which is justified as the series and integral involved are finite, the left hand side equals

$$\frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(u_1, \cdots, u_r) \prod_{k=1}^{r} \phi_k(u_k) z_k^{u_k} \frac{1}{\prod_{j=1}^{r} \Gamma(e_j + \sum_{i=1}^{r} \alpha_j^{(i)} u_i)}$$

$$\times \sum_{s_1=0}^{k_1} \sum_{t_1=0}^{l_1} \cdots \sum_{s_r=0}^{k_r} \sum_{t_r=0}^{l_r} \frac{(-k_1)_{s_1}(-l_1)_{t_1} \cdots (-k_r)_{s_r}(-l_r)_{t_r}}{s_1!t_1! \cdots s_r!t_r!}$$

$$\times \frac{(g_1)_{s_1}(h_1)_{t_1} \cdots (g_r)_{s_r}(h_r)_{t_r}}{(g_1 - h_1 - k_1 + 1)_{s_1}(h_1 - g_1 - l_1 + 1)_{t_1} \cdots (g_r - h_r - k_r + 1)_{s_r}(h_r - g_r - l_r + 1)_{t_r}}$$

$$\times \frac{\prod_{j=1}^{r} (e_j + \sum_{i=1}^{r} \alpha_j^{(i)} u_i)_{s_j+t_j}}{\prod_{j=1}^{r} (e_j + \sum_{i=1}^{r} \alpha_j^{(i)} u_i)_{s_j} \prod_{j=1}^{r} (e_j + \sum_{i=1}^{r} \alpha_j^{(i)} u_i)_{t_j}} \, \mathrm{d}u_1 \cdots \mathrm{d}u_r \quad (2.2)$$

Now making use of the following known result due to Carlitz [[4], 6, p.139]

$$\sum_{s_1=0}^{k_1} \sum_{t_1=0}^{l_1} \frac{(-k_1)_{s_1}(-l_1)_{t_1}}{s_1!t_1!} \frac{(g_1)_{s_1}(h_1)_{t_1}}{(g_1 - h_1 - k_1 + 1)_{s_1}(h_1 - g_1 - l_1 + 1)_{t_1}}$$

$$= \frac{(g_1)_{l_1}(h_1)_{k_1}(e_1)_{k_1+l_1}}{(g_1 - h_1)_{l_1}(h_1 - g_1)_{k_1}(e_1)_{k_1}(e_1)_{l_1}} \quad (2.3)$$

in the above expression (2.2), we get the right hand side of the equation (2.1).

3. Special cases

If $\tau_i = \tau_{i^{(k)}} = 1$, the Aleph-function of several variables degenere in the I-function of several variables defined by Sharma and Ahmad [6]. We obtain the following relation.

$$\sum_{s_1=0}^{k_1} \sum_{t_1=0}^{l_1} \cdots \sum_{s_r=0}^{k_r} \sum_{t_r=0}^{l_r} \frac{(-k_1)_{s_1}(-l_1)_{t_1} \cdots (-k_r)_{s_r}(-l_r)_{t_r}}{s_1!t_1! \cdots s_r!t_r!}$$

$$\frac{(g_1)_{s_1}(h_1)_{t_1} \cdots (g_r)_{s_r}(h_r)_{t_r}}{(g_1 - h_1 - k_1 + 1)_{s_1}(h_1 - g_1 - l_1 + 1)_{t_1} \cdots (g_r - h_r - k_r + 1)_{s_r}(h_r - g_r - l_r + 1)_{t_r}}$$

$$I_{p_i+r,q_i+2r;R:W}^{0,\mathfrak{n}+r:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right|\begin{array}{c}(1\text{ - e}_1-s_1-t_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-s_r-t_r;\alpha'_r,\cdots,\alpha_r^{(r)}),\\ \cdots\\ (1\text{ - e}_1-s_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-s_r;\alpha'_r,\cdots,\alpha_r^{(r)}),\end{array}$$

$$\left.\begin{array}{c}\text{A' : C'}\\ \cdots\\ (1\text{ - e}_1-t_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-t_r;\alpha'_r,\cdots,\alpha_r^{(r)}),B':D'\end{array}\right)$$

$$=\frac{(g_1)_{l_1}(h_1)_{k_1}\cdots(g_r)_{l_r}(h_r)_{k_r}}{(g_1-h_1)_{l_1}(h_1-g_1)_{k_1}\cdots(g_r-h_r)_{l_r}(h_r-g_r)_{k_r}}$$

$$I_{p_i+r,q_i+2r;R:W}^{0,\mathfrak{n}+r:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right|\begin{array}{c}(1\text{ - e}_1-k_1-l_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-k_r-l_r;\alpha'_r,\cdots,\alpha_r^{(r)}),\\ \cdots\\ (1\text{ - e}_1-k_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-k_r;\alpha'_r,\cdots,\alpha_r^{(r)}),\end{array}$$

$$\left.\begin{array}{c}\text{A' : C'}\\ \cdots\\ (1\text{ - e}_1-l_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-l_r;\alpha'_r,\cdots,\alpha_r^{(r)}),B':D'\end{array}\right) \tag{3.1}$$

Provided that :

$$|argz_k|<\frac{1}{2}A_i'^{(k)}\pi\,,\ \text{where : }A_i'^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0\text{ , with }k=1\text{ to }r\text{ , }i=1\text{ to }R\text{ , }i^{(k)}=1\text{ to }R^{(k)} \tag{3.2}$$

If $R=R^{(1)}=,\cdots,R^{(r)}=1$, the multivariable I-function degenere in the multivariable H-function defined by Srivastava et al [8] and we have :

$$\sum_{s_1=0}^{k_1}\sum_{t_1=0}^{l_1}\cdots\sum_{s_r=0}^{k_r}\sum_{t_r=0}^{l_r}\frac{(-k_1)_{s_1}(-l_1)_{t_1}\cdots(-k_r)_{s_r}(-l_r)_{t_r}}{s_1!t_1!\cdots s_r!t_r!}$$

$$\frac{(g_1)_{s_1}(h_1)_{t_1}\cdots(g_r)_{s_r}(h_r)_{t_r}}{(g_1-h_1-k_1+1)_{s_1}(h_1-g_1-l_1+1)_{t_1}\cdots(g_r-h_r-k_r+1)_{s_r}(h_r-g_r-l_r+1)_{t_r}}$$

$$H_{p+r,q+2r:W}^{0,\mathfrak{n}+r:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right|\begin{array}{c}(1\text{ - e}_1-s_1-t_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-s_r-t_r;\alpha'_r,\cdots,\alpha_r^{(r)}),\\ \cdots\\ (1\text{ - e}_1-s_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-s_r;\alpha'_r,\cdots,\alpha_r^{(r)}),\end{array}$$

$$\left.\begin{array}{c} \text{A''} : \text{C''} \\ \cdots \\ (1 - \mathrm{e}_1 - t_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-t_r;\alpha'_r,\cdots,\alpha_r^{(r)}),B'':D'' \end{array}\right)$$

$$= \frac{(g_1)_{l_1}(h_1)_{k_1}\cdots(g_r)_{l_r}(h_r)_{k_r}}{(g_1-h_1)_{l_1}(h_1-g_1)_{k_1}\cdots(g_r-h_r)_{l_r}(h_r-g_r)_{k_r}}$$

$$H^{0,\mathfrak{n}+r:V}_{p+r,q+2r:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\middle| \begin{array}{c} (1 - \mathrm{e}_1 - k_1 - l_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-k_r-l_r;\alpha'_r,\cdots,\alpha_r^{(r)}), \\ \cdots \\ (1 - \mathrm{e}_1 - k_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-k_r;\alpha'_r,\cdots,\alpha_r^{(r)}), \end{array}\right.$$

$$\left.\begin{array}{c} \text{A''} : \text{C''} \\ \cdots \\ (1 - \mathrm{e}_1 - l_1;\alpha'_1,\cdots,\alpha_r^{(r)}),\cdots,(1-e_r-l_r;\alpha'_r,\cdots,\alpha_r^{(r)}),B'':D'' \end{array}\right) \tag{3.3}$$

Provided that :

$$|argz_i| < \frac{1}{2}A_i\pi\ , \quad \text{where} : A_i = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(i)} - \sum_{j=\mathfrak{n}+1}^{p}\alpha_j^{(i)} - \sum_{j=1}^{q}\beta_j^{(i)} + \sum_{j=1}^{n_i}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}\gamma_j^{(i)} + \sum_{j=1}^{m_i}\delta_j^{(i)}$$

$$- \sum_{j=m_i+1}^{q_i}\delta_j^{(i)} > 0\ , \text{ with } i = 1,\cdots,r \tag{3.4}$$

If $r = 2$, we obtain the H-function of two variables defined by Srivastava et al [8].

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s}{r!r!}\frac{(b-e-n+1)_r(b)_s}{(2-e-m-n)_r(e)_s}H^{0,\mathfrak{n}+1:V}_{p+1,q+2:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \mathrm{z}_2 \end{array}\middle| \begin{array}{c} (1 - \text{c-r-s};\alpha,\beta),A_1;C_1, \\ \cdots \\ (1 - \text{c-r};\alpha,\beta),(1-c-s;\alpha,\beta),B_1;D_1 \end{array}\right)$$

$$= \frac{(b)_m(e-b)_n}{(e)_n(e+n-1)_m}H^{0,\mathfrak{n}+1:V}_{p+1,q+2:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \mathrm{z}_2 \end{array}\middle| \begin{array}{c} (1 - \text{c-m-n};\alpha,\beta),A_1;C_1, \\ \cdots \\ (1 - \text{c-m};\alpha,\beta),(1-c-n;\alpha,\beta),B_1;D_1 \end{array}\right) \tag{3.5}$$

This double finite expansion formula is obtained by Gupta and Goyal [5].

4. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as I-function of several variables defined by Sharma and Ahmad [6] , multivariable H-function , see Srivastava et al [8] , the Aleph-function of two variables defined by K.sharma [7], the I-function of two variables defined by Goyal and Agrawal [1,2,3] , ,and the h-function of two variables , see Srivastava et al [8].

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