ADDITIVE – QUARTIC FUNCTIONAL EQUATIONS ARE STABLE IN QUASI-BANACH SPACE

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Abstract. In this paper, the authors established the solution and generalized Ulam - Hyers stability of the additive-quartic functional equation

$$f\left[2x_1+\sum_{i=2}^{n}x_i\right]+f\left[2x_1-\sum_{i=2}^{n}x_i\right]=4\left[f\left(2x_1+\sum_{i=2}^{n}x_i\right)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)\right]$$
$$-3\left[f\left(\sum_{i=2}^{n}x_i\right)+f\left(-\sum_{i=2}^{n}x_i\right)\right]+10f(x_1)+14f(-x_1)$$

in Quasi Banach spaces .

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1. INTRODUCTION

The study of perturbation problems for functional equations is related to a famous question of S.M. Ulam [22] concerning the stability of group homomorphisms. It was affirmatively answered by Hyers [12] for Banach spaces. It was further generalized and interesting results obtained by number of mathematicians ([2], [8], [17], [20], [21]). For more detailed information about such problems one can see ([2]-[5], [7], [9], [13]-[16], [18]).

In this paper, the authors established the solution and generalized Ulam - Hyers stability of the additive-quartic functional equation of the form

$$f\left[2x_1+\sum_{i=2}^{n}x_i\right]+f\left[2x_1-\sum_{i=2}^{n}x_i\right]=4\left[f\left(2x_1+\sum_{i=2}^{n}x_i\right)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)\right]$$
$$-3\left[f\left(\sum_{i=2}^{n}x_i\right)+f\left(-\sum_{i=2}^{n}x_i\right)\right]+10f(x_1)+14f(-x_1) \qquad (1.1)$$

in Quasi Banach spaces using direct method.

2. GENERAL SOLUTION OF (1.1)

In this section, we present the solution of the functional equation (1.1). Through out this section let X and Y be real vector spaces.

Theorem 2.1 An odd function $f:X\rightarrow Y$ satisfies the functional equation (1.1) then f is additive.

Proof. Let $f:X\rightarrow Y$ satisfies the functional equation (1.1). Letting $(x_1,x_2,.......,x_n)$ by $(0,0,.......,0)$ in (1.1), we get $f(0)=0$. Replacing $(x_1,x_2,.......,x_n)$ by $(x,0,.......,0)$ and $(x,x,.......,x)$ in (1.1) respectively, and using oddness of f, we obtain $f(2x)=2f(x), f(3x)=3f(x)$, for all $x\in X$. Replacing $(x_1,x_2,.......,x_n)$ by $(x,y,0,.....,0)$ and using oddness in (1.1), we get

$$f(2x+y)+f(2x-y)=4\left[f(x+y)+f(x-y)\right]+4f(x) \qquad (2.1)$$

Letting $(x+y,x-y)$ by (u,v) in (2.1), we obtain

$$f(x+u)+f(x+v)=4\left[f(u)+f(v)\right]-4f(x) \qquad (2.2)$$

Replacing (u,v) by (y,y) in (2.2), we obtain

$$2f(x+y)=8f(y)-4f(x) \qquad (2.3)$$

Intrechanging x and y, we get

$$2f(x+y)=8f(x)-4f(y) \tag{2.4}$$

Adding (2.3) and (2.4), we obtain

$$f(x+y)=f(x)+f(y)$$

Hence the equation (1.1) is additive.

Lemma 2.2 An even function $f:X\to Y$ satisfies the functional equation (1.1) then f is quartic.

Proof. Let $f:X\to Y$ satisfies the functional equation (1.1). Using evenness of f and replacing $(x_1,x_2,.......,x_n)$ and $(x,y,0,.....,0)$, we get

$$f(2x+y)+f(2x-y)=4\left[f(x+y)+f(x-y)\right]+12[f(x)+f(-x)]$$
$$-3[f(y)+f(-y)]-2[f(x)-f(-x)].$$

It is clear that f is quartic [16].

3. STABILITY RESULTS OF (1.1): DIRECT METHOD

Throughout this section, let us consider E_1 is a Quasi-Banach space with quasi-norm $\|.\|_{E_1}$ and E_2 is a p- Banach space with p -norm. $\|.\|_{E_2}$. Let K be the modulus of concavity of $\|.\|_{E_2}$. Define a mapping $f:E_1\to E_2$ by

$$\begin{aligned} Df(x_1,x_2,.....,x_n)=& f\left[2x_1+\sum_{i=2}^{n}x_i\right]+f\left[2x_1-\sum_{i=2}^{n}x_i\right]-4\left[f\left(2x_1+\sum_{i=2}^{n}x_i\right)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)\right] \\ &+3\left[f\left(\sum_{i=2}^{n}x_i\right)+f\left(-\sum_{i=2}^{n}x_i\right)\right]-10f(x_1)-14f(-x_1) \end{aligned} \tag{3.1}$$

for all $x_i\in E_1$, $i=1,2,....,n$ and we state the following Lemma 3.1 [15] without proof, it will be useful in proving our theorems.

Lemma 3.1

Let $0<p\le 1$ and let $x_1,x_2,........,x_n$ be non negative real numbers then

$$\left(\sum_{i=1}^{n}x_i\right)^p\le\left(\sum_{i=1}^{n}x_i^p\right). \tag{3.2}$$

Theorem 3.2

Let $\phi:\underbrace{E_1\times E_1\times,........\times E_1}_{n\ times}\to[0,\infty)$ be a function such that for all $x_i\in E_1$, $i=1,2,....,n$

$$\lim_{n\to\infty}(16)^n\,\phi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},....,\frac{x_n}{2^n}\right)=0 \tag{3.3}$$

and

$$\sum_{i=1}^{\infty}(16)^{ip}\phi^p\left(\frac{x_1}{2^i},\frac{x_2}{2^i},....,\frac{x_n}{2^i}\right)<\infty \tag{3.4}$$

for all $x_i\in E_1$ and for all $x_1,x_2\in\{0,x\}$, $x_i\in\{0\}$, where $i=3,4,....,n.$ Suppose that an even function $f:E_1\to E_2$ with $f(0)=0$ satisfies the inequality

$$\|Df(x_1,x_2,.....,x_n)\|_{E_2}\le\phi(x_1,x_2,.....,x_n)\quad\forall x_i\in E_1 \tag{3.5}$$

Then the limit

$$Q(x)=\lim_{n\to\infty}(16)^n f\left(\frac{x}{2^n}\right) \tag{3.6}$$

exists for all $x\in E_1$ and $Q:E_1\to E_2$ is a unique quartic function satisfying

$$\|f(x)-Q(x)\|_{E_2}\le\frac{k}{16}\left[\psi_e(x)\right]^{\frac{1}{p}}\ ,\quad\forall x\in E_1 \tag{3.7}$$

where

$$\psi_e(x) = \sum_{i=1}^{\infty} \frac{(16)^{ip}}{2^p} \left\{ \phi^p\left(\frac{x}{2^i}, 0, 0, ..., 0\right) + \phi^p\left(0, \frac{x}{2^i}, 0, ..., 0\right) \right\}$$

for all $x \in E_1$.

Proof. Using evenness of f and replacing $(x_1, x_2, ..., x_n)$ by $(x, y, 0, ..., 0)$ in (3.5) ,we get

$$\left\| f(2x+y) + f(2x-y) - 4\left[f(x+y) + f(x-y)\right] - 24f(x) + 6f(y) \right\|_{E_2} \le \phi(x, y, 0, ..., 0) \quad (3.8)$$

for all $x, y \in E_1$ Replacing (x, y) by (y, x) in (3.8) and using evenness, we have obtain

$$\left\| f(x+2y) + f(x-2y) - 4\left[f(x+y) + f(x-y)\right] - 24f(y) + 6f(x) \right\|_{E_2} \le \phi(y, x, 0, ..., 0) \quad (3.9)$$

for all $x, y \in E_1$,from (3.8) and (3.9) and replacing y by 0, we have

$$\left\| f(2x) - 16f(x) \right\|_{E_2} \le \frac{k}{2}\left[\phi(x, 0, 0, ..., 0) + \phi(0, x, 0, ..., 0)\right] \quad , \qquad \forall x \in E_1.$$

which can be written as

$$\left\| f(2x) - 16f(x) \right\|_{E_2} \le \frac{k}{2}\psi(x) \quad , \qquad \forall x \in E_1, \qquad (3.10)$$

and

$$\psi(x) = \frac{1}{2}\left[\phi(0, x, 0, ..., 0) + \phi(0, x, 0, ..., 0)\right], \ \forall x \in E_1, \qquad (3.11)$$

in equation (3.10), replace x by $\frac{x}{2^{n+1}}$ and multiplying both sides by $(16)^n$, we have

$$\left\| (16)^{n+1} f\left(\frac{x}{2^{n+1}}\right) - (16)^n f\left(\frac{x}{2^n}\right) \right\|_{E_2} \le k(16)^n \psi\left(\frac{x}{2^{n+1}}\right) \quad , \qquad \forall x \in E_1 \ , \qquad (3.12)$$

for all non-negative integers n , since $x \in E_2$ is a p-Banach space and using (3.12) , we obtain

$$\begin{aligned} \left\| (16)^{n+1} f\left(\frac{x}{2^{n+1}}\right) - (16)^m f\left(\frac{x}{2^m}\right) \right\|_{E_2}^p &\le \sum_{i=m}^{n} \left\| (16)^{i+1} f\left(\frac{x}{2^{i+1}}\right) - (16)^i f\left(\frac{x}{2^i}\right) \right\|_{E_2}^p \\ &\le k^p \sum_{i=m}^{n} 16^{ip} \psi^p\left(\frac{x}{2^{i+1}}\right) \end{aligned} \qquad (3.13)$$

for all non-negative integers n and m with $n \ge m$ and all $x \in E_1$. Now $0 < p \le 1$ and with the help of Lemma 3.1, the equation (3.11) can be written as

$$\psi^p(x) = \frac{1}{2^p}\left[\phi^p(x, 0, 0, ..., 0) + \phi^p(0, x, 0, ..., 0)\right], \qquad \forall x \in E_1. \qquad (3.14)$$

Therefore it follows from (3.4) and (3.14) that

$$\sum_{i=1}^{\infty} 16^{ip} \psi^p\left(\frac{x}{2^i}\right) < \infty \ , \forall x \in E_1. \qquad (3.15)$$

Therefore, we conclude from (3.13) and (3.15) that the sequence $\left\{(16)^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in E_1$, since E_2 is complete, the sequence $\left\{(16)^n f\left(\frac{x}{2^n}\right)\right\}$ converges for all $x \in E_1$. Now we define the mapping by $Q: E_1 \to E_2$ by (3.6) for all $x \in E_1$. Letting $m = 0$ and allowing $n \to \infty$ in (3.13) , we get

$$\left\| f(x) - Q(x) \right\|_{E_2}^p \le k^p \sum_{i=0}^{\infty} (16)^{ip} \psi^p\left(\frac{x}{2^{i+1}}\right) = \frac{k^p}{(16)^p} \sum_{i=0}^{\infty} \psi^p\left(\frac{x}{2^i}\right) \quad , \qquad \forall x \in E_1. \qquad (3.16)$$

Use (3.11) in the equation (3.16), we arrive at the result (3.7). Now, we show that Q is a quartic it follows from (3.3), (3.5) and (3.6),

$$\|DQ(x_1,x_2,\dots,x_n)\|_{E_2} = \lim_{n\to\infty}16^n \left\|Df\left(\frac{x_1}{2^n},\frac{x_2}{2^n},\dots,\frac{x_n}{2^n}\right)\right\|_{E_2} \le 16^n \phi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},\dots,\frac{x_n}{2^n}\right), \quad \forall x_1,x_2,..,x_n \in E_1.$$

Therefore the mapping $Q: E_1 \to E_2$ satisfies (1.1). Since $Q(x)=0$, then by Theorem 2.1, we obtain that the mapping $Q: E_1 \to E_2$ is quartic. To prove the uniqueness of Q, let $Q^{'}: E_1 \to E_2$ be another quartic mapping satisfying (3.7). Since

$$\lim_{n\to\infty}16^n \sum_{i=1}^{\infty}16^{ip}\phi^p\left(\frac{x_1}{2^{n+i}},\frac{x_2}{2^{n+i}},\dots,\frac{x_n}{2^{n+i}}\right) = \lim_{n\to\infty}16^{ip}\phi^p\left(\frac{x_1}{2^i},\frac{x_2}{2^i},\dots,\frac{x_n}{2^i}\right) = 0, \forall x_1,x_2,..,x_n \in E_1,$$

and for all $x_1, x_2 \in \{0,x\}, x_i \in \{0\}$ where $i=3,4,\dots,n$ then

$$\lim_{n\to\infty}(16)^{np}\psi_e\left(\frac{x}{2^n}\right)=0, \qquad \forall x \in E_1. \tag{3.17}$$

It follows from (3.7) and (3.17),

$$\left\|Q(x)-Q^{'}(x)\right\|_{E_2}^p = \lim_{n\to\infty}16^n \left\|f\left(\frac{x}{2^n}\right)-Q^{'}\left(\frac{x}{2^n}\right)\right\|_{E_2}^p \le \left(\frac{k}{16}\right)^p \lim_{n\to\infty}16^{np}\psi\left(\frac{x}{2^n}\right)=0, \ \forall x \in E_1,$$

so $Q=T$. Hence the theorem is proved .

Theorem 3.3 Let $\phi: \underbrace{E_1\times E_1\times,\dots\dots\times E_1}_{n\ times} \to [0,\infty)$ be a function such that for all $x_i \in E_1$, $i=1,2,\dots,n$ $\lim_{n\to\infty}\frac{1}{(16)^n}\phi\left(2^n x_1, 2^n x_2,\dots, 2^n x_n\right)=0, \forall x_1,x_2,\dots x_n \in E_1$

and $\displaystyle\sum_{i=1}^{\infty}\frac{1}{(16)^{ip}}\phi^p\left(2^i x_1, 2^i x_2,\dots,2^i x_n\right)<\infty$

for all $x_i \in E_1$, $i=1,2,\dots,n$, for all $x_1, x_2 \in \{0,x\}, x_i \in \{0\}$, where $i=3,4,\dots,n$. Suppose that an even function $f: E_1 \to E_2$ with $f(0)=0$ satisfies the inequality (3.5) for all $x_i \in E_1$, $i=1,2,\dots,n$

Then the limit $Q(x)=\lim_{n\to\infty}\frac{1}{16^n}f\left(2^n x\right)$

Exists for all $x \in E_1$ and $Q: E_1 \to E_2$ is a unique quartic function satisfying

$$\| f(x)-Q(x)\|_{E_2} \le \frac{k}{16}[\psi_e(x)]^{\frac{1}{p}}, \qquad \forall x \in E_1, \tag{3.18}$$

where $\psi_e = \sum_{i=0}^{\infty}\frac{1}{2^p 16^{ip}}\left\{\phi^p(2^i x,0,0,\dots,0)+\phi^p(0,2^i x,0,\dots,0)\right\}$

for all $x \in E_1$.

Proof. The proof of the theorem is similar to that of Theorem 3.2 .

Corollary 3.4 Let λ, r be non negative real numbers such that $r<4$, suppose that an even function $f: E_1 \to E_2$ which satisfies the inequality

$$\|D f(x_1,x_2,\dots,x_n)\|_{E_2} \le \lambda\left[\sum_{i=1}^{n}\|x_i\|_{E_1}^{r}\right], \ \forall x_i \in E_1. \tag{3.19}$$

Then there exists a unique quartic function $Q: E_1 \to E_2$ satisfies

$$\| f(x)-Q(x)\|_{E_2} \le \frac{k}{32}\left[\frac{1}{\left|1-2^{(r-4)p}\right|}\|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \ \forall x \in E_1$$

The proof of the above corollary is similar to that of Theorem 3.2 for f is even.

Theorem 3.5 Let $\phi: \underbrace{E_1\times E_1\times,\dots\dots\times E_1}_{n\ times} \to [0,\infty)$ be a function such that for all $x_i \in E_1$

where $i=1,2,\dots,n$, $\lim_{n\to\infty}2^n\phi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},\dots,\frac{x_n}{2^n}\right)=0$ and

$$\sum_{i=1}^{\infty}2^{ip}\phi\left(\frac{x_1}{2^i},\frac{x_2}{2^i},\dots,\frac{x_n}{2^i}\right)<\infty, \qquad \forall x_1,x_2,\dots,x_n \in E_1$$

and for all $x_1, x_2 \in \{0, x\}$ and $x_i \in \{0\}$ where $i = 3, 4, ..., n.$ Suppose that an odd function $f : E_1 \to E_2$ satisfies the inequality (3.5) . Then the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in E_1$ and $A : E_1 \to E_2$ is a unique additive function satisfying

$$\| f(x) - A(x)\|_{E_2} \le \frac{k}{2}\left[\varphi_o(x)\right]^{\frac{1}{p}} \qquad \forall x \in E_1$$

Where $\varphi_o(x) = \sum_{i=1}^{\infty} \left(2^{i-1}\right)^p \left\{ \phi^p\left(\frac{x}{2^i}, 0, 0, ..., 0\right) + \phi^p\left(0, \frac{x}{2^i}, 0, ..., 0\right)\right\}$ for all $x \in E_1$.

The proof of the above Theorem is similar to that of Theorem 3.2 for f is odd.

Theorem 3.6. Let $\phi : \underbrace{E_1 \times E_1 \times, ... \times E_1}_{n\ times} \to [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{1}{2^n} \phi\left(2^n x_1, 2^n x_2, ..., 2^n x_n\right) = 0, \quad \forall x_1, x_2, ..., x_n \in E_1, \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{2^{ip}} \phi^p\left(2^i x_1, 2^i x_2, ..., 2^i x_n\right) < \infty$$

for all $x_i \in E_1$, $i = 1, 2, ..., n$, for all $x_1, x_2 \in \{0, x\}$, $x_i \in \{0\}$, where $i = 3, 4, ..., n$. Suppose that an odd function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality (3.5). Then the limit $A(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ for all $x \in E_1$ and $A : E_1 \to E_2$ is a unique additive function satisfying

$$\| f(x) - A(x)\|_{E_2} \le \frac{k}{2}\left[\varphi_o(x)\right]^{\frac{1}{p}} \qquad \forall x \in E_1$$

Where $\varphi_o(x) = \sum_{i=1}^{\infty} \frac{1}{\left(2^{i+1}\right)^p} \left\{ \phi^p(2^i x, 0, 0, ..., 0) + \phi^p(0, 2^i x, 0, ..., 0) \right\}$ for all $x \in E_1$.

The proof of the above Theorem is similar to that of Theorem 3.3 for f is odd.

Corollary 3.7. Let λ be non negative real number and r be a real number such that $r < 1$, suppose that an odd function $f : E_1 \to E_2$ which satisfies the inequality (3.19). Then there exists a unique additive function $A : E_1 \to E_2$ satisfies

$$\| f(x) - A(x) \|_{E_2} \le \frac{k}{4}\left[\frac{1}{\left|1 - 2^{(4-r)p}\right|} \|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \ \forall x \in E_1.$$

The proof of the following theorem is similar to that of Theorem 3.5 for f is odd. Hence the details of the proof are omitted.

Theorem 3.8. Let $\phi : E_1 \times E_1 \times, ... \times E_1 \to [0, \infty)$ be a function satisfies (3.3) and (3.4) for all $x_i \in E_1$, and for all $x_1, x_2 \in \{0, x\}$, $x_i \in \{0\}$ where $i = 3, 4, ..., n$. Suppose that a function $f : E_1 \to E_2$ satisfies the inequality (3.5) with $f(0) = 0$ for all $x \in E_1$, then there exists a unique quartic function $Q : E_1 \to E_2$ and a unique additive function $A : E_1 \to E_2$ satisfies (1.1) and

$$\| f(x) - Q(x) - A(x)\|_{E_2} \le \frac{k^3}{32}\left\{\left[\psi_e(x) + \psi_e(-x)\right]^{\frac{1}{p}} + 8\left[\varphi_o(x) + \varphi_o(-x)\right]^{\frac{1}{p}}\right\}, \quad \forall x \in E_1 \tag{3.20}$$

where $\psi_e(x)$ and $\varphi_o(x)$ has been defined in Theorem 3.3 and 3.5 respectively, for all $x \in E_1$.

Proof. We have $f_e(x) = \frac{f(x) + f(-x)}{2}$ for all $x \in E_1$. Therefore $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$\left\|Df_e(x_1, x_2, ..., x_n)\right\| = \frac{k}{2}\left[\phi\left(x_1, x_2, ..., x_n\right) + \phi\left(-x_1, -x_2, ..., -x_n\right)\right] \quad , \quad \forall x_1, x_2, ..., x_n \in E_1.$$

Let

$$\mu(x_1, x_2, ..., x_n) = \frac{k}{2}\left[\phi\left(x_1, x_2, ..., x_n\right) + \phi\left(-x_1, -x_2, ..., -x_n\right)\right] \quad , \quad x_i \in E_1 \text{ where } i = 1, 2, ..., n. \tag{3.21}$$

Then using Lemma 3.1, we obtain

$$\mu^p(x_1,x_2,...,x_n) \le \left(\frac{k}{2}\right)^p \left[\phi^p\left(x_1,x_2,...,x_n\right)+\phi^p\left(-x_1,-x_2,...,-x_n\right)\right] \quad , \qquad \forall x_i \in E_1 .$$

Then therefore

$$\sum_{i=1}^{\infty} (16)^{ip} \mu^p\left(\frac{x_1}{2^i},\frac{x_2}{2^i},...,\frac{x_n}{2^i}\right) \le \infty \quad , \qquad \forall x_i \in E_1 ,$$

and for all $x_1, x_2 \in \{0, x\}, x_i \in \{0\}$ where $i = 3,4,...,n$. Hence, in view of Theorem 3.2, there exists a unique quartic function $Q: E_1 \to E_2$ satisfies

$$\| f_e(x) - Q(x)\|_{E_2} \le \frac{k}{16}\left[\mu_e(x)\right]^{\frac{1}{p}}, \qquad \forall x \in E_1 \tag{3.22}$$

where

$$\mu_e(x) = \sum_{i=1}^{\infty} \frac{16^{ip}}{2^p}\left\{\phi^p\left(\frac{x}{2^i},\frac{x}{2^i},\frac{0}{2^i},...,\frac{0}{2^i}\right)+\phi^p\left(\frac{0}{2^i},\frac{x}{2^i},...,\frac{0}{2^i}\right)\right\}, \ \forall x_i \in E_1. \tag{3.23}$$

Applying (3.21) in (3.23) and using the Lemma 3.1, we obtain

$$\left\| f(x) - Q(x) \right\| = \left(\frac{k}{2}\right)^p \left[\psi_e(x) + \psi_e(-x)\right], \qquad \forall x_i \in E_1 . \tag{3.24}$$

Therefore it follows from (3.21),

$$\| f(x) - Q(x)\|_{E_2} \le \frac{k^2}{32}\left[\psi_e(x) + \psi_e(-x)\right]^{\frac{1}{p}}, \qquad \forall x \in E_1 \tag{3.25}$$

Also we have $f_o(x) = \dfrac{f(x) - f(-x)}{2}$ for all $x \in E_1$. Therefore $f_o(0) = 0$, $f_o(-x) = -f_o(x)$ and $\left\|Df_o(x_1,x_2,...,x_n)\right\| \le \mu\left(x_1,x_2,...,x_n\right)$, $\forall x_i \in E_1$. From Theorem 3.5, it follows that there exists a unique additive function $A: E_1 \to E_2$ satisfies

$$\| f_o(x) - A(x)\|_{E_2} \le \frac{k}{2}\left[\eta_o(x)\right]^{\frac{1}{p}}, \qquad \forall x \in E_1 \tag{3.26}$$

where

$$\eta_o(x) = \sum_{i=1}^{\infty}\left(2^{i-1}\right)^p \left\{\phi^p\left(\frac{x}{2^i},0,0,...,0\right) + \phi^p\left(0,\frac{x}{2^i},0,...,0\right)\right\} \tag{3.27}$$

Using the above ideas as given in (3.24), we arrive

$$\eta_o(x) = \left(\frac{k}{2}\right)^p \left[\varphi_o(x) + \varphi_o(-x)\right] \tag{3.28}$$

Again using (3.28) in (3.25), we obtain

$$\| f_o(x) - A(x)\|_{E_2} \le \left(\frac{k}{2}\right)^p \left[\phi_o(x) + \phi_o(-x)\right]^{\frac{1}{p}}, \qquad \forall x \in E_1$$

Then the result (3.20) follows from (3.25) and (3.28).

Theorem 3.9. Let $\phi: E_1 \times E_1 \times, ... \times E_1 \to [0,\infty)$ be a function satisfies (3.3) and (3.4) for all $x_i \in E_1$, and for all $x_1, x_2 \in \{0, x\}, x_i \in \{0\}$ where $i = 3,4,....,n$. Suppose that a function $f: E_1 \to E_2$ satisfies the inequality (3.5) with $f(0) = 0$ for all $x \in E_1$, then there exists a unique quartic function $Q: E_1 \to E_2$ and a unique additive function $A: E_1 \to E_2$ satisfies (1.1) and

$$\| f(x)-Q(x)-A(x)\|_{E_2} \leq \frac{k^3}{32}\left\{\left[\psi_e(x)+\psi_e(-x)\right]^{\frac{1}{p}}+8\left[\varphi_o(x)+\varphi_o(-x)\right]^{\frac{1}{p}}\right\}, \quad \forall x \in E_1$$

where $\psi_e(x)$ and $\varphi_o(x)$ has been defined in Theorem 3.3 and 3.5 respectively, for all $x \in E_1$.
Proof. The proof of this Theorem follows from Theorem 3.3 and Theorem 3.7 and it is very similar to the Theorem 3.8 and so the proof is omitted here.

Corollary 3.10. Let λ be non negative real number and r be a real number such that $r < 4$, suppose that an function $f: E_1 \to E_2$ which satisfies the inequality (3.20) then there exists a unique quartic function $Q: E_1 \to E_2$ and a unique additive function $A: E_1 \to E_2$ satisfies (1.1) then

$$\| f(x)-Q(x)-A(x)\|_{E_2} \leq \frac{k}{2^{p+1}}\left[\left\{\frac{1}{8\left|1-2^{(r-4)p}\right|}+\frac{1}{\left|1-2^{(r-1)p}\right|}\right\}\|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \quad \forall x \in E_1 \tag{3.29}$$

Proof. Define the function $\phi: E_1 \times E_1 \times, \ldots\ldots.. \times E_1 \to [0,\infty)$ by $\phi(x_1, x_2, \ldots\ldots, x_n) \leq \lambda\left[\sum_{i=1}^{n}\|x_i\|_{E_1}^{r}\right]$.

Using the Corollary 3.4, we obtain

$$\| f(x)-Q(x)\|_{E_2} \leq \frac{k}{8\left(2^{p+1}\right)}\left[\frac{1}{\left|1-2^{(r-4)p}\right|}\|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \quad \forall x \in E_1 \tag{3.30}$$

again using Corollary 3.7, we obtain

$$\| f(x)-A(x)\|_{E_2} \leq \frac{k}{2^{p+1}}\left[\frac{1}{\left|1-2^{(r-1)p}\right|}\|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \quad \forall x \in E_1 \tag{3.31}$$

Adding the equations (3.30) and (3.31), we obtain (3.29). Hence the proof.

REFERENCES

[1] Aczel J. and Dhombres J., Functional Equations in Several Variables, Cambridge Univ, Press, 1989.
[2] Aoki T., On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64-66.
[3] Arunkumar M., Rassias John M., On the generalized Ulam-Hyers stability of an AQ-mixed type functional equation with counter examples, Far East Journal of Applied Mathematics, Volume 71, No. 2, (2012), 279-305.
[4] Arunkumar M., Agilan P., Additive Quadratic functional equation are stable in Banach space: A Fixed Point Approach, International Journal of pure and Applied Mathematics, Vol. 86 No.6, 951-963, (2013).
[5] Arunkumar M., Agilan P., Additive Quadratic functional equation are stable in Banach space: A Direct Method, Far East Journal of Mathematical Sciences, Volume 80, No. 1, (2013), 105 – 121.
[6] S. S. Chang, Y. J. Cho, and S. M. Kang, Nonlinear Operator Theory in Probabilistic Metric Spaces, Nova Science Publishers, Huntington, NY, USA, 2001.
[7] Czerwik S., Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.
[8] Gavruta P., A generalizationof the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184 (1994), 431-436.
[9] Eshaghi Gordji M., Ghobadipour N., Rassias J. M., Fuzzy stability of additive quadratic functional Equations, arXiv:0903.0842v1 [math.FA] 4 Mar 2009.

[10] Hadzic O., Pap E., Fixed Point Theory in Probabilistic Metric Spaces, vol. 536 of Mathematics and its Applications, Kluwer Academic, Dordrecht, The Netherlands, 2001.
[11] Hadzic O., Pap E. and Budincevic M., Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces, Kybernetika, vol. 38, no. 3, (2002) 363-382.
[12] Hyers D. H., On the stability of the linear functional equation, Proc. Nat. Acad.Sci.,U.S.A., 27, (1941), 222-224.
[13] Hyers D. H., Isac G., Rassias Th. M., Stability of unctional equations in several variables, Birkhauser, Basel, 1998.
[14] Jun K. W., Kim H. M., On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation, Bull. Korean Math. Soc. 42(1), (2005), 133-148.
[15] Jung S. M., Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.
[16] Lee S.H., Im S.M., Hwang I.S., Quartic functional equations, J. Math. Anal. Appl., 307, (2005), 387-394.
[17] Kannappan Pl., Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, 2009.
[18] Murthy S., Arunkumar M., Ganapathy G., Rajarethinam P., Stability of mixed type additive quadratic functional equation in Random Normed space, International Journal of Applied Mathematics Vol. 26. No. 2 (2013), 123-136.
[19] Najati A., Moghimi M. B., On the Stability of a quadratic and additive functional equation, J. Math. Anal. Appl. 337 (2008), 399-415.
[20] Park C., Orthogonal Stability of an Additive-Quadratic Functional Equation, Fixed Point Theory and Applications 2001 2011:66.
[21] Rassias M. J., Arunkumar M., Ramamoorthi S., Stability of the Leibniz additive-quadratic functional equation in Quasi-Beta normed space: Direct and fixed point methods, Journal of Concrete and Applicable Mathematics (JCAAM), (Accepted).
[22] Ulam S. M., Problems in Modern Mathematics, Science Editions, Wiley, New York, 1964.