Strong limiting regions of hypertension vector sequences of order statistics

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Abstract

This paper finds a limiting region for the patient's hypertension behavior based on several values above the cut off value of blood pressure taking together , by properly normalizing the vector sequences comprising of moving maxima ($Y_{k(n)}$) , moving second maxima ($S_{k(n)}$) ,…, moving K^{th} maxima $K_{k(n)}$, K being the K^{th} order statistics , using Borel Cantelli lemma, which is meaningful in taking the decision on hypertension for treatment and so on. However, for ease of computation, results are proved for $Y_{k(n)}$ and $S_{k(n)}$, under certain conditions on F and k(n). The results are just stated for K order statistics.

Keywords: Almost sure limit set; Moving maxima; Moving second maxima; Vector sequence

1. Introduction

Under clinical trial set up in medical field , to study the hypertensive patient, he must be under observation for several days. Patient's hypertension behavior based on several values above the cut off value of blood pressure taking together is meaningful in taking the decision on hypertension for treatment and so on. Thus the hypertension values on the jth day are independent. This scenario demands a new model for the limiting behavior by considering the K^{th} order statistics of hypertension values of a patient. When observations are more, leaving first few observations do not matter a much. Thus the K^{th} order statistics values of a patient on the j^{th} day are a moving order statistics. As a result, this paper finds a limiting region for above scenario by properly normalizing the vector sequences comprising of moving maxima ($Y_{k(n)}$) , moving second maxima ($S_{k(n)}$) ,…, moving M^{th} maxima $M_{k(n)}$, M being the M^{th} order statistics , using Borel Cantelli lemma. However, for ease of computation, results are proved for $Y_{k(n)}$ and $S_{k(n)}$ for the following set up. The results are stated for M moving order statistics.

Let $\{X_n , n\geq 1\}$ be a sequence of independent identically distributed random variables (i.i.d.r.v) with common distribution function (d.f) F. Define, moving maxima $Y_{k(n)} = \max(X_{n+1} , X_{n+2} , …., X_{n+ k(n)})$ where k(n) is a sequence of positive integers , $2\leq k(n) \leq n$. The term moving maxima is due to Rothmann and Russo (1991). In the light of the concept of moving maxima, define moving second maxima as $S_{k(n)}$ = second $\max(X_{n+1} , X_{n+2} , …., X_{n+ k(n)})$ and moving third maxima as $T_{k(n)}$ = Third $\max(X_{n+1} , X_{n+2} , …., X_{n+ k(n)})$ for the same k(n) and so on. For $k(n) \Xi n$, $Y_{k(n)} \Xi Y_n$, $S_{k(n)} \Xi S_n$, $T_{k(n)} \Xi T_n$ and so on.

De Hann and Hordijk (1972) have obtained lim sup and lim inf of properly normalized Y_n , under the following assumptions on F.

Assumption A(C). F has positive derivative F′(x) for all sufficiently large x and $\lim_{n\to\infty} \frac{g(x)}{x} = c, 0 < c < \infty$

where $g(x) = \frac{1-F(x)}{F'(x)} \log_2\left(\frac{1}{1-F(x)}\right)$ with $\log_2$ u= loglog u.

Assumption B. F is twice differentiable and F′(x) is positive for all sufficiently large x with $\lim_{n\to\infty} g'(x) = 0$.

Hebbar and Vadiraja(1996-97, 97) generalized the Gut's (1990) result using moving maxima and Borel_Cantelli (B_C) lemma approach for more general K(n):

K(n) is non-decreasing (1.1)

$$Sup[k(n+1) - k(n)] \le \mu \ (finite) \quad (1.2)$$

and

$$k(n) = \left[\frac{n}{(\log n)^{t(n)}}\right] \text{ where } t(n) \to \infty \text{ as } n \to \infty . \quad (1.3)$$

Next, define b_n through $1\text{-}F(b_n) = n^{-1}$ and they made

Assumption C. $1 - F(b_n x) = \frac{(\log n)^{\gamma(1-\beta(x))}}{n^{\beta(x)}} a_n(x), x > 0$ where γ is a constant, β(x) is strictly increasing and positive, $\lim_{n\to\infty} a_n(x) = \theta(x) > 0$.

For the standard normal distribution, Assumption C holds with $\beta(x)=x^2$, $a_n(x)= c_n(x)\theta(x)$, $\theta(x) = \left[\frac{(4\pi)^{x^2/2}}{x\sqrt{2}}\right]$,

$c_n(x) \to 1$ as $n\to \infty$, γ=-1/2, using the fact that $1-F(x) \sim \left[\frac{e^{-x^2/2}}{x\sqrt{2\pi}}\right]$ as $n\to\infty$ and for the unit exponential distribution β(x)=x, $a_n(x)=1$, γ=0.

When c=∞, the assumption is as follows.

Assumption D. Let the support F be [0,∞) and let $h(x) = \frac{g(x)}{x}$.

$$1 - F(b_n x^{h(b_n)}) = \frac{(\log n)^{\gamma(n, x^{h(b_n)})}}{n}, \forall x > 0 \quad (1.4)$$

where $\lim_{n\to\infty} r(n, x^{h(b_n)}) = -\log x$. (1.5)

Under Assumption D, the following example holds good.

Example 1. 1-F(x) =1/x, x≥1 . Then, b_n = n, h(x)=loglogx and hence $r(n, x^{h(b_n)}) = -\log x$.

Example 2. 1-F(x) =1/log x, x≥e . Then, $b_n = e^{-n}$, h(x)=logx loglogx and hence

$1 - F(b_n x^{h(b_n)}) = \dfrac{1}{n(1+\log x \log\log x)}$ which does not satisfy Assumption C.

Throughout, δ_i's, i=1,2,.. are sufficiently small positive constants. Now the results are stated below.

Theorem 1. Under Assumption A(C), the almost sure limit set of the vector sequence

$\{ Y_{k(n)}/b_n , S_{k(n)}/b_n \}$ n≥1, coincides with the region $S_1=\{(x,y):\ e^{-pc} \le x \le e^c ,\ e^{-pc} \le y \le e^c ,\ x \ge y,$

$xy \le e^{(1-p)c}\}$ $0 \le p < \infty$.

Theorem 2. Under the conditions of Theorem 1 but with p=∞, the almost sure limit set of the vector sequence $\{ Y_{k(n)}/b_n \}$ n≥1, coincides with the region $S_2=[0,e^c]$, provided there exists a strictly decreasing sequence $q_n \sim$ k(n)/n and either

i. $k(n) \ge \gamma \log n$ kn) ≥ γ log n for some γ>0 and

ii. 1< k(n) ≤ γ log n

Theorem 3. Under the condition of Theorem 1 but with p = ∞, $\lim_{n\to\infty} (S_{k(n)}/b_n) = 0$ … a.s.

Theorem4. Under Assumption B, the almost sure limit set of the vector sequence

$\{ (Y_{k(n)}-b_n)/d_n , (S_{k(n)}-b_n)/d_n \}$ n≥1, coincides with the region $S_3=\{(x,y):\ -p \le x \le 1 ,\ -p \le y \le 1 ,\ x \ge y,$

$x+y \le (1-p)\}$ $0 \le p < \infty$, where $d_n = f(b_n) \log_2 n$ with f(x)= (1-F(x))F'(x).

When p=∞, Theorem 4 can be refined as

Theorem5. Under Assumption B and Assumption C, the almost sure limit set of the vector sequence $\{ Y_{k(n)}/b_n ,$ $S_{k(n)}/b_n \}$ n≥1, coincides with the region $S_4=\{(x,y):\ d \le x \le 1 ,\ d \le y \le 1 ,\ x \ge y,$

$\beta(x)+\beta(y) \le 1+\Delta\}$, where d satisfies $d= \beta^{-1}(\Delta)$ for all $\epsilon > 0$ with $\Delta= \lim_{n\to\infty} [\log k(n)/\log n]$

Now, the corollaries follow

Corollary 1. Under the conditions of Theorem1, the almost sure limit set of the sequence

i. $\{Y_{k(n)}/b_n \}$n≥1, coincides with the region $S_5=[e^{-pc}, e^c]$

ii. $\{S_{k(n)}/b_n \}$n≥1, coincides with the region $S_6 = [e^{-pc}, e^{(1-p)c/2}]$, $0 \le p < \infty$.

Corollary 2. Under the conditions of Theorem4, the almost sure limit set of the sequence

i. $\{ (Y_{k(n)}-b_n)/d_n \}$ n≥1, coincides with $S_7 = [-p, 1-p]$

ii. $\{ (S_{k(n)}-b_n)/d_n \}$ n≥1, coincides with $S_8 = [-p, (1-p)/2]$, $0\leq p<\infty$.

Corollary 3. Under the conditions of Theorem5 but with $p=\infty$,the almost sure limit set of the sequence

i. $\{Y_{k(n)}/b_n \}$n≥1, coincides with the region $S_9=[d,1]$

ii. $\{S_{k(n)}/b_n \}$n≥1, coincides with the region $S_{10} = [d, \beta^{-1} ((1+\Delta)/2)]$

Remark: Let $Y^*_{k(n)} = \max(X_{n-k(n)+1} , X_{n-k(n)+2} , \ldots, X_n)$ and $S^*_{k(n)}$ = second $\max(X_{n-k(n)+1} , X_{n-k(n)+2} , \ldots, X_n)$ are the backward moving maxima and backward moving second maxima respectively. Then the above results hold good.

2. Proofs.

The proof of Theorem 1 is built up through the following lemmas.

Lemma 1.2. (Ortega and Wschebor 1984, Lemma 1).

Let (A_n) n≥1, be a sequence of events on a probability space. If

$$\sum_{n=1}^{\infty} P(A_n) = \infty \tag{2.1}$$

and

$$\liminf_{n\to\infty} \frac{\sum_{1\leq i<j\leq n}\sum \left(P(A_i \cap A_j) - P(A_i)P(A_j)\right)}{\left(\sum_{i=1}^{n} P(A_i)\right)^2} \leq 0 \tag{2.2}$$

then P(A_n i.o)=1.

Lemma 2.2. (De Haan and Hordijk 1972, pp 1190-92)

$$1 - F(b_n x) = \frac{(\log n)^{\gamma(n,x)}}{n}, \forall x > 0 \tag{2.3}$$

where

$$\lim_{n\to\infty} r(n,x) = \frac{-\log x}{c} \text{ under Assumption A} \tag{2.4}$$

and

$$\lim_{n\to\infty} r(n,x) = -x \text{ under Assumption B} \tag{2.5}$$

where $b_n x$ in (2.3) is replaced by $d_n x + b_n$.

Lemma 3.2. For every Є>0, $x>y>e^{-pc}$ and $xy<e^{(1-p)c}$,

$$P\left(Y_{k(l_i)} > (x+\varepsilon)b_{l_i}, S_{k(l_i)} > y b_{l_i} \ i.o\right) = 0 \tag{2.6}$$

and

$$P\left(Y_{k(l_i)} > x b_{l_i}, S_{k(l_i)} > (y+\varepsilon) b_{l_i} \ i.o\right) = 0 \tag{7.2}$$

where $l_i = \left\lfloor e^{i^\theta} \right\rfloor$, and $\theta = \left[2p + \frac{\log xy}{c} + \frac{\log(1+\varepsilon)}{4c}\right]^{-1}$

Proof. Notice that by (3.2)

$$P\left(Y_{k(l_i)} > (x+\varepsilon)b_{l_i}, S_{k(l_i)} > y b_{l_i}\right)$$

$$= P\left(Y_{k(l_i)} > (x+\varepsilon)b_{l_i}\right) - P\left(Y_{k(l_i)} > (x+\varepsilon)b_{l_i}, S_{k(l_i)} \le y b_{l_i}\right)$$

$$= 1 - F^{k(l_i)}\left((x+\varepsilon)b_{l_i}\right) - k(l_i) F^{k(l_i)-1}\left(y b_{l_i}\right).\left(1 - F\left((x+\varepsilon)b_{l_i}\right)\right)$$

$$= \frac{(1+0(1))}{2} k(l_i).(k(l_i)-1)\left(1 - F\left((x+\varepsilon)b_{l_i}\right)\right).\left(1 - F\left(y b_{l_i}\right)\right)$$

$$\le \text{Const. } k^2(l_i).\left(1 - F\left((x+\varepsilon)b_{l_i}\right)\right).\left(1 - F\left(y b_{l_i}\right)\right) \tag{8.2}$$

for all i large, since,

$$k(l_i).\left(1 - F\left((x+\varepsilon)b_{l_i}\right)\right) \to 0, \text{ for } x > e^{-pc}$$

$$k(l_i).\left(1 - F\left(y b_{l_i}\right)\right) \to 0, \text{ for } y > e^{-pc}$$

and

$$\left[\frac{\left(1 - F\left((x+\varepsilon)b_{l_i}\right)\right)}{\left(1 - F\left(y b_{l_i}\right)\right)}\right] \to 0, \text{ for } x > y.$$

$$= \text{Const.} \left(\frac{k(l_i)}{l_i}\right)^2 .(\log l_i)^{r(l_i, x+\varepsilon)+r(l_i, y)} = E_i, \text{ (say)}$$

Now, as i→∞, in view of (3.1) and (4.2),

$$\left(2t(l_i) - r(l_i, x+\varepsilon) + r(l_i, y)\right) \to \left(2p + \frac{\log xy}{c} + \frac{\log(1+\varepsilon)}{c}\right)$$

Hence, for every Є>0 and for I large, we have

$$(\log l_i)^{-\left(2p + \frac{\log xy}{c} + \frac{3\log(1+\varepsilon)}{2c}\right)} < E_i < (\log l_i)^{-\left(2p + \frac{\log xy}{c} + \frac{\log(1+\varepsilon)}{2c}\right)} \tag{9.2}$$

$$\theta.\left(2t(l_i) - r(l_i, x+\varepsilon) + r(l_i, y)\right) > \theta.\left(2p + \frac{\log xy}{c} + \frac{\log(1+\varepsilon)}{2c}\right) = 1 + \delta_1$$

(10.2)

where $\delta_1 = \left[\frac{7\theta \log(1+\varepsilon)}{4c}\right] > 0$

by (2.9).

Then, in view of (8.2) and (10.2),

$$\sum P\left(Y_{k(l_i)} > (x+\varepsilon)b_{l_i}, S_{k(l_i)} > yb_{l_i}\right) < \infty$$

Accordingly, by B_C lemma, (6.2) follows.

The Proof of (7.2) is similar.

Lemma 4.2. For $e^{-pc}<x<e^{c}$, $e^{-pc}<y<e^{c}$, x>y and $xy<e^{(1-p)c}$,

$$P\left(Y_{k(l_i)} > xb_{l_i}, S_{k(l_i)} > yb_{l_i}\ i.o\right) = 1$$

Proof. Similar to that at (8.2) and then at (10.2),

$$P\left(Y_{k(l_i)} > xb_{l_i}, S_{k(l_i)} > yb_{l_i}\right)$$

$$= \text{Const. } k(l_i).(k(l_i)-1).\left(1-F\left(xb_{l_i}\right)\right).\left(1-F\left(yb_{l_i}\right)\right)$$

$$= \text{Const. } i^{-\theta.(2t(l_i) - r(l_i, x) - r(l_i, y))}$$

=Const. $i^{-(1+\delta_2)}$ (11.2)

where $\delta_2 = \left[\frac{\theta \log(1+\varepsilon)}{12c}\right] > 0$ for every Є>0 and i large.

To establish the claim made, it is sufficient to show the events $P\left(Y_{k(l_i)} > xb_{l_i}, S_{k(l_i)} > yb_{l_i}\right)$, i≥1 are independent for all i large.

$$l_i - k(l_i) + 1 - l_{i-1} = l_i\left[1 - \frac{k(l_i)}{l_i} + \frac{1}{l_i} - \frac{l_{i-1}}{l_i}\right] \tag{12.2}$$

Now observe that

i. By (3.1), $(\log l_i)^{-(p+\delta)} < \frac{k(l_i)}{l_i} < (\log l_i)^{-(p-\delta)}$, for δ>0, i large.

ii. For θ>1, $\frac{l_{i-1}}{l_i} \to 0$ as i→∞

iii. For θ≤1, $\frac{l_{i-1}}{l_i} \to 1$, $1 - \frac{l_{i-1}}{l_i} \sim h i^{\theta-1}$ as i→∞ where h is a positive constant.

Hence, whenever θ>1, i.e. $\left(2p + \frac{\log xy}{c}\right) < 1$,

R.H.S(12.2) is ~ l_i as i→∞. Further for $(1+p)^{-1} < \theta \le 1$, the expression inside the square bracket of (12.2) is ~ $hi^{(\theta-1)}$ as i→∞, since $i^{(1-\theta)} . \frac{k(l_i)}{l_i} \to 0$.

Thus, for $\theta > (1+p)^{-1}$, i.e. for $xy < e^{(1-p)c}$,

R.H.S(12.2) tends to ∞ as i→∞.

Thus, the events under consideration are independent, for all i large.

Lemma 5.2 . For all $x \ge e^{-pc}$, $y \ge e^{-pc}$ with $xy > e^{(1-p)c}$ and for every Є>0,

$$P\left(Y_{k(n)} > (x+\varepsilon)b_n, S_{k(n)} > (y+\varepsilon)b_n \ i.o\right) = 0$$

Proof. Define the events

$$A_n = \left\{Y_{k(n)} > (x+\varepsilon)b_n, S_{k(n)} > (y+\varepsilon)b_n\right\} \text{ and}$$

$$B_i = \left\{Y_{k(n)} > (x+\varepsilon)b_i, S_{k(n)} > (y+\varepsilon)b_i \ \text{ for at least one } n \in [n_i, n_{i+1}) \right\}$$

where $n_i = \left[e^{i^\theta} \right]$, i≥1, $\theta = \left[2p + \frac{\log xy}{c} \right]^{-1}$

Notice that

$$P(A_n.i.o \text{ in n}) \le P(B_i.i.o \text{ in i}) \tag{13.2}$$

But, P(B_i) ≤

$$P\left(\max(X_{n_i-k(n_i)+1,},\ldots,X_{n_{i+1}}) > (x+\varepsilon)b_{n_i}.\text{second}\max(X_{n_i-k(n_i)+1,},\ldots,X_{n_{i+1}}) > (y+\varepsilon)b_{n_i}\right)$$

$$\le \left(\frac{n_{i+1} - n_i + k(n_i) - 1}{n_i} \right)^2 .(\log n_i)^{r(n_i, x+\varepsilon)+r(n_i, y+\varepsilon)} \tag{14.2}$$

by (3.2).

But , for θ<$(1+p)^{-1}$, i.e. for xy>$e^{(1-p)c}$ and in view of (3.1),

$$\left(\frac{n_{i+1} - n_i + k(n_i) - 1}{n_i} \right) \le \frac{Const.}{i^{\theta.p}} \tag{15.2}$$

In view of (15.2) and (4.2), (14.2) becomes

$$\le \frac{Const}{i^{\theta.\left(2p+\frac{\log xy}{c}\right)+2\theta\left(\frac{\log(1+\varepsilon/x)+\log(1+\varepsilon/y)}{c}\right)}} \le \frac{Const}{i^{1+\delta_3}}$$

for every Є>0 and i large, where $\delta_3 = \left[\frac{2\theta \log(1+\varepsilon/x) + \log(1+\varepsilon/y)}{c} \right] > 0$.

By appealing to B_C lemma, $P(B_i \ i.o) = 0$.

This completes the proof of our lemma via (13.2).

Lemma 6.2 . For every Є>0 and $x_0 = e^{-pc}(1-Є)$

$$P\left(S_{k(n)} > x_0 b_n \ i.o\right) = 0$$

Proof. This is accomplished by showing

$$P\left(S_{k(n)} \le x_0 b_n\right) \to 0 \tag{16.2}$$

and

$$\sum_{n=1}^{\infty} P\left(S_{k(n)} \le x_0 b_n \text{ and } S_{k(n+1)} > x_0 b_{n+1}\right) < \infty \tag{17.2}$$

Note that,

$$P\left(S_{k(n)} \le x_0 b_n\right) = P\left(S_{k(n)} \le x_0 b_n, Y_{k(n)} \le x_0 b_n\right) + P\left(S_{k(n)} \le x_0 b_n, Y_{k(n)} > x_0 b_n\right)$$

$$= P\left(Y_{k(n)} \le x_0 b_n\right) + P\left(S_{k(n)} \le x_0 b_n, Y_{k(n)} > x_0 b_n\right)$$

$$= F^{k(n)}\left(x_0 b_n\right) + k(n) F^{k(n)-1}\left(x_0 b_n\right).\left(1 - F\left(x_0 b_n\right).\right)$$

$$= E_n F^{k(n)-1}\left(x_0 b_n\right).\left(1 + \frac{F\left(x_0 b_n\right)}{E_n}\right)$$

$$\le E_n F^{k(n)-1}\left(x_0 b_n\right).\left(1 + \frac{1}{E_n}\right) \tag{18.2}$$

where $E_n = k(n).\left(1 - F\left(x_0 b_n\right)\right)$.

By (3.1), (3.2), (4.2) we have for every Є>0 and for some a>0

$$(\log n)^{a-\varepsilon} < E_n < (\log n)^{a+\varepsilon} \text{ for all large n.} \tag{19.2}$$

Fix M>0, so that M(a-Є)- (a-Є)>1+δ_4, δ_4>0.

By (3.2) and (19.2),

$$\text{R.H.S}(2.18) \le (1+o(1))\, e^{-(\log n)^{a-\varepsilon}} (\log n)^{a+\varepsilon} \le (1+o(1))\, (\log n)^{(1+\delta_4)} \tag{20.2}$$

Thus, in view of (20.2) and (18.2), (16.2) holds.

Now, since b_n is non decreasing, notice that

$$P\left(S_{k(n)} \le x_0 b_n \text{ and } S_{k(n+1)} > x_0 b_{n+1}\right)$$

$$\le P\left(S_{k(n)} \le x_0 b_n, Y_{k(n)} \le x_0 b_n, S_{k(n+1)} > x_0 b_n\right) + P\left(S_{k(n)} \le x_0 b_n, Y_{k(n)} > x_0 b_n, S_{k(n+1)} > x_0 b_n\right) \tag{21.2}$$

Case(i). When n-k(n+1)+2≥n-k(n)+1

$$\text{R.H.S}(2.21) \le k(n) F^{k(n)-1}\left(x_0 b_n\right).\left(1 - F\left(x_0 b_n\right).\right)^2 \tag{22.2}$$

By (3.2),(4.2) and (19.2) and on similar lines to (20.2), for all n large and δ_5>0,

$$\text{R.H.S}(22.2) \le \frac{Const}{n(\log n)^{1+\delta_5}} \tag{23.2}$$

Thus, in view of (21.2) and (23.2), (17.2) holds.

Case(ii). When n-k(n+1)+2<n-k(n)+1, the arrangement of observation is as follows.

$X_{n-k(n+1)+2}, \ldots, X_{n-k(n)}, X_{n-k(n)+1} \ldots, X_n, X_{n+1}$

Thus, R.H.S (2.21)

= P(At least two among $X_{n-k(n+1)+2}, \ldots, X_{n-k(n)}, X_{n+1} > x_0 b_n$). P(Each of $X_{n-k(n)+1} \ldots, X_n$ is $\le x_0 b_n$) +P(At least one among $X_{n-k(n+1)+2}, \ldots, X_{n-k(n)}, X_{n+1} > x_0 b_n$). P(Exactly one among $X_{n-k(n)+1} \ldots, X_n > x_0 b_n$ and Remaining (k(n)-1) observations are $\le x_0 b_n$)

={1- P(At the most one among $X_{n-k(n+1)+2}, \ldots, X_{n-k(n)}, X_{n+1} \le x_0 b_n$)}. $F^{k(n)}(x_0 b_n) +$

{1- P(All (k(n+1)-k(n)) observations are $\le x_0 b_n$)}. $k(n) F^{k(n)-1}(x_0 b_n)[1 - F(x_0 b_n)]$

$$= F^{k(n)}(x_0 b_n).\left[1 - F^{k(n+1)-k(n)}(x_0 b_n) - (k(n+1)-k(n)).(1-F(x_0 b_n)).F^{k(n+1)-k(n)-1}(x_0 b_n)\right] +$$

$$k(n) F^{k(n)}(x_0 b_n).[1 - F(x_0 b_n)].\left[1 - F^{k(n+1)-k(n)}(x_0 b_n)\right]$$

$$\le F^{k(n)-1}(x_0 b_n).\left[1 - F^{k(n+1)-k(n)}(x_0 b_n) - (k(n+1)-k(n)).(1-F(x_0 b_n)).F^{k(n+1)-k(n)-1}(x_0 b_n)\right.$$

$$\left. + k(n)(1-F(x_0 b_n))(1 - F^{k(n+1)-k(n)}(x_0 b_n))\right]$$

Using the fact that $e^{-t} = 1 = t(1+o(1))$, where $t=(1-F(x_0 b_n)) \to 0$, as $n \to \infty$,

$$\le F^{k(n)-1}(x_0 b_n)\left[(k(n+1)-k(n)).(1-F(x_0 b_n)).(1+o(1)) + k(n)(k(n+1)-k(n)).(1-F(x_0 b_n))^2.(1+o(1))\right]$$
$$\le (1+o(1)).k(n)(k(n+1)-k(n)).F^{k(n)-1}(x_0 b_n).(1-F(x_0 b_n))^2.$$

in view of (18.2), for all large n.

On similar lines to case (i), (17.2) holds.

Lemma 7.2 . No point in the region

$\{e^{-pc} \le x \le e^c, e^{-pc} \le y \le e^c \text{ with } x < y.\}$ is a limit point of $\left\{\frac{Y_{k(n)}}{b_n}, \frac{S_{k(n)}}{b_n}\right\} n \ge 1.$

Proof. For any subsequence {n_l}, since $Y_{k(n_l)} > S_{k(n_l)}$ a.s., we should have x≥y which is a contradiction.

Hence the lemma.

The following lemma is trivial , hence the details are omitted.

Lemma 8.2 . For every Є>0 and $x_0 = e^{-pc}$

$$P\left(Y_{k(n)} < (x_0 - \varepsilon) b_n \, i.o\right) = 0$$

Proof of Theorem 1. S_1 is a required limit set by lemmas 5.2, 6.2, 7.2 and 8.2 . We conclude with the fact that the limit set is necessarily closed and from the lemmas 3.2 and 4.2. This completes the proof of Theorem 1.

Proof of Theorem 2. On similar lines to Theorem 1 and hence details are skipped.

When p=∞, the results for $S_{k(n)}$ are as follows.

Lemma 9.2 $\limsup_{n\to\infty} \frac{S_{k(n)}}{b_n} \le 0 \quad a.s$

Proof. This is accomplished by showing for every Є>0,

$$P\left(S_{k(n)} > \varepsilon b_n \, i.o\right) = 0 \tag{24.2}$$

This is in turn will follow, when we show

$$P\left(S_{k(n)} > \varepsilon b_n\right) \to 0 \text{ as } n\to\infty \tag{25.2}$$

and $$\sum_{n=1}^{\infty} P\left(S_{k(n)} \le \varepsilon b_n \; and S_{k(n+1)} > \varepsilon b_{n+1}\right) < \infty \tag{26.2}$$

Next, (25.2) and (26.2) are accomplished on similar lines to lemma 6.2 and hence through (24.2), lemma 9.2 is achieved.

Lemma 10.2 . $\liminf_{n\to\infty} \frac{S_{k(n)}}{b_n} \ge 0 \quad a.s$

Proof. This is trivial and hence the details are omitted.

Proof of Theorem 3. We conclude that the limit exists for $S_{k(n)}$ from lemmas 9.2 and 10.2. Hence the proof of Theorem.

Due to Theorem 3, the almost sure limit set of vector sequence $\left\{\frac{Y_{k(n)}}{b_n}, \frac{S_{k(n)}}{b_n}\right\} n \ge 1$, shrinks to the interval $\left[0, e^{c}\right]$ on the x axis.

Proof of Theorem 4. Similar to Theorem 1, choosing $l_i = \left\lfloor e^{i^{\theta}} \right\rfloor$, and $\theta = \left[2p + x + y + \frac{\varepsilon}{2} \right]^{-1}$.

Proof of Theorem 5. Similar to Theorem 1, but with $l_i = \left[i^{\theta} \right]$ where $\theta = \left[\beta(x) + \beta(y) - 2\Delta + \frac{\varepsilon}{2} \right]^{-1}$

Following are the results for Mth order statistics**.**

Theorem 6. Under Assumption A(C), the almost sure limit set of the vector sequence

$\{ Y_{k(n)}/b_n , S_{k(n)}/b_n \ \dots, M_{k(n)}/b_n \}$ $n\geq1$, coincides with the region $S_1=\{(x,y, \dots, m)$: $e^{-pc}\leq x\leq e^c$, $e^{-pc}\leq y\leq e^c$, …., $e^{-pc}\leq m\leq e^c$, $x \geq y\geq\dots\geq m$, $xy\dots m\leq e^{(1-p)c}\}$ $0\leq p<\infty$.

Theorem 7. Under the conditions of Theorem 1 but with $p=\infty$, the almost sure limit set of the vector sequence $\{ Y_{k(n)}/b_n \}$ $n\geq1$, coincides with the region $S_2=[0,e^c]$, provided there exists a strictly decreasing sequence $q_n \sim k(n)/n$ and either

i. $k(n) \geq \gamma \log n$ kn) $\geq \gamma$ log n for some $\gamma>0$ and

ii. $1< k(n) \leq \gamma \log n$

and

$\lim_{n\to\infty} (S_{k(n)}/b_n) = 0$, …, $\lim_{n\to\infty} (M_{k(n)}/b_n) = 0$ … a.s.

Theorem 8. Under Assumption B, the almost sure limit set of the vector sequence

$\{ (Y_{k(n)}-b_n)/d_n , (S_{k(n)}-b_n)/d_n , \dots,(M_{k(n)}-b_n)/d_n \}$ $n\geq1$, coincides with the region $S_3=\{(x,y,.. m)$: $-p\leq x\leq 1$, $-p\leq y\leq 1$, … , $p\leq m\leq 1$, $x \geq y\geq\dots\geq m$,

$x+y+\dots+m\leq (1-p)\}$ $0\leq p<\infty$, where $d_n = f(b_n) \log_2 n$ with $f(x)= (1-F(x))F'(x)$.

When $p=\infty$, Theorem 4 can be refined as

Theorem9. Under Assumption B and Assumption C, the almost sure limit set of the vector sequence $\{ Y_{k(n)}/b_n , S_{k(n)}/b_n \ \dots, M_{k(n)}/b_n \}$ $n\geq1$, coincides with the region $S_4=\{(x,y, \dots, k)$: $d\leq x\leq 1$, $d\leq y\leq 1$, …, $d\leq m\leq 1$, $x \geq y \geq \dots,\geq m$,

$\beta(x)+\beta (y) + \dots +\beta (m)\leq 1+\Delta\}$, where d satisfies $d= \beta^{-1}(\Delta)$ for all $\epsilon > 0$ with $\Delta= \lim_{n\to\infty} [\log k(n)/\log n]$

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