Differential Transform Method to Solve Linear and Non Linear Diffusion Equation

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Abstract: *In present paper differential Transform Method (DTM) has been employed to solve the linear and non-linear differential diffusion equation. The result obtain by DTM shows a good agreement with the exact solution obtained by the existing method in literature.*

Keywords — *Differential Transform Method, Linear & Nonlinear Diffusion Differential Equation.*

I. INTRODUCTION

The mathematical modelling of physical phenomena arising in many disciplines either result into ordinary differential equation, system of differential equation, partial differential equation or system of partial differential equation with initial and boundary condition. In literature many methods of exact approximate, numerical and analytical methods are available which are computationally intensive or need linearization, discretization. To overcome all this difficulties Differential Transform Method is applied on linear and nonlinear diffusion equation.

Diffusion equation are used to describes density dynamics in a material undergoing diffusion. The equation is usually written as

$$\frac{\partial \phi(\mathrm{x},t)}{\partial t} = \nabla \ D(\phi,\mathrm{x})\nabla\phi(\mathrm{x},t)$$

Where (x, t) is the density of the diffusing material at distance x and time t and $D(\phi, r)$ is the diffusion coefficient for the density at distance. If the diffusion coefficient depends on the density the equation is non-linear otherwise it is linear. Abdul-Majid Wazwaz in 2001 solve the non-linear diffusion equation with mainly power law diffusivities by using Adomain Decomposition Method [6]. Vibrational Iterative Method was implemented to find the exact solution of non-linear fast and slow diffusion equation by A.Sadighi, D. D. Ganji[4]. Homotopy Perturbation Method was successfully applied to solve linear and non-linear diffusion equation by Khayti Desai and V.H.Pradhan[7]. M.A.AL. Jawary applied new iterative method (NIM or DJM) proposed by Daftardar-Gejji and Jafari to find the solution of linear and non –linear wave and diffusion equation arising in mathematical physics and engineering field. In this paper we solve the following type of diffusion equation with initial condition:

$$u_t = u_{xx} + f(x), 0 < x < L$$

$$u_t = u_{xx} + g(x,t), 0 < x < L$$

$$u_t = (D(u)u_x)_x, 0 < x < L$$

With initial condition and boundary condition

$$u(x,0) = \mathrm{f(x)}$$

$$u(x,t) = g(x)$$

The functions f (u) and g (x, t) are linear and source functions respectively. The function D(u) is the diffusion term that plays an important role in a wide range of applications in diffusion processes. The diffusion term D(u) appears in several forms such as power law and exponential forms.

This paper is organized in the following manner. In paper section 1 describes the two dimensional differential transform method to solve the non-linear as well as linear partial differential equation. Numerical example is solved using Differential Transform method in section 2. Conclusion is presented in section 3.

II. DIFFERENTIAL TRANSFORM METHOD

Differential Transform Method is semi numerical method which is derive from Taylors methods that convert the given differential equation to recursive formula which is used to calculate the coefficient of the series.

Definition 1: Two-Dimensional Differential transform of the function w(x,y) is defined as follows[1,2,3]

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial y^h} w(x,y)\right]$$

In (1), w (x, y) is the original function and W (k, h) is the transformed f which is called T function.

Definition 2: Differential Inverse Transform of W (k, h) is defined as follows:

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^{k}y^{h} \qquad (2)$$

Using (1) and (2), we obtain

$$W(k,h)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^{k}\partial y^{h}}w(x,y)\right]_{\substack{x=0\\y=0}}x^{k}y^{h}.$$

Table 1: Fundamental Operation of DTM[1,2,3]

Original Function	Transform Function
$u(x,y)\pm v(x,y)$	$W(h,k)=U(k,h)\pm V(k,h)$
$\alpha v(x,y)$	$W(k,h)=\lambda U(k,h)$
$w(x,y)=\dfrac{\partial u(x,y)}{\partial x}$	$(k+1)U(k+1,h)$
$w(x,y)=\dfrac{\partial u(x,y)}{\partial y}$	$(h+1)U(k,h+1)$
$w(x,y)=\dfrac{\partial^{r+s}u(x,y)}{\partial x^{r}\partial y^{s}}$	$(k+1)(k+2)\ldots\ldots(k+r)(h+1)(h+2)$ $\ldots\ldots(h+s)U(k+r,h+s)$
$u(x,y)=v(x,y)w(x,y)$	$\sum_{r=0}^{k}\sum_{s=0}^{h}V(r,h-s)W(k-r,s)$
$x^{m}y^{n}$	$\delta(k-m,h-n)=\begin{cases}1 & k=m \text{ and } h=n\\ 0 & \text{otherwise}\end{cases}$
$e^{\lambda(x+y)}$	$\dfrac{\lambda^{(k+h)}}{k!h!}$
$x^{m}\sin(ax+b)$	$\dfrac{a^{h}}{h!}\delta(k-m)\sin\left(\dfrac{h\pi}{2}+b\right)$
$x^{m}e^{ay}$	$\dfrac{a^{h}}{h!}\delta(k-m)$

III. Numerical Examples

In this section, we apply Differential Transform form method to the linear and nonlinear in homogenous, homogenous and heat transfer equation. Diffusion.

Example 1: Consider equation[6]

$$u_{t}=u_{xx}-u,\quad 0<x<\pi,t>0 \qquad (3)$$

With initial and boundary condition

$$u(x,0)=\sin x \qquad (4)$$

$$u(0,t)=0 \qquad (5)$$

$$u(\pi,t)=0 \qquad (6)$$

Applying DTM to (1) we get the following recursive formula

$$(h+1)U(k,h+1)=(k+2)(k+1)U(k+2,h)-U(k,h) \qquad (7)$$

Applying DTM to (2), (3) we get

$$U(k,0)=\begin{cases}0 & \text{if } k \text{ even}\\ \dfrac{1}{k!} & k=4n-3,n\in N\\ \dfrac{(-1)^{k}}{k!} & k=4n-1,n\in N\end{cases} \qquad (8)$$

$$U(0,h)=0,h=0,1,2,\ldots. \qquad (9)$$

Using (7), (8), (9) we get following coefficient table

k	h=0	h=11	h=2	h=3	h=4	h=5	h=6	h=7	h=8
0	0	0	0	0	0	0	0	0	0
1	$\frac{1}{0!1!}$	$\frac{-2}{1!1!}$	$\frac{2^{2}}{1!2!}$	$\frac{-2^{3}}{1!3!}$	$\frac{2^{4}}{1!4!}$	$\frac{-2^{5}}{1!5!}$	$\frac{2^{6}}{1!6!}$	$\frac{-2^{7}}{1!7!}$	$\frac{2^{8}}{1!8!}$
2	0	0	0	0	0	0	0	0	0
3	$\frac{-1}{0!3!}$	$\frac{2}{3!1!}$	$\frac{-2^{2}}{3!2!}$	$\frac{2^{3}}{3!3!}$	$\frac{-2^{4}}{3!4!}$	$\frac{2^{5}}{3!5!}$	$\frac{-2^{6}}{3!6!}$	$\frac{2^{7}}{3!7!}$	$\frac{-2^{8}}{3!8!}$
4	0	0	0	0	0	0	0	0	0
5	$\frac{1}{0!5!}$	$\frac{-2}{5!1!}$	$\frac{2^{2}}{5!2!}$	$\frac{-2^{3}}{5!3!}$	$\frac{2^{4}}{5!4!}$	$\frac{-2^{5}}{5!5!}$	$\frac{2^{6}}{5!6!}$	$\frac{-2^{7}}{5!7!}$	$\frac{2^{8}}{5!8!}$
6	0	0	0	0	0	0	0	0	0
7	$\frac{-1}{0!7!}$	$\frac{2}{7!1!}$	$\frac{-2^{2}}{7!2!}$	$\frac{2^{3}}{7!3!}$	$\frac{-2^{4}}{7!4!}$	$\frac{2^{5}}{7!5!}$	$\frac{-2^{6}}{7!6!}$	$\frac{2^{7}}{7!7!}$	$\frac{-2^{8}}{7!8!}$
8	0	0	0	0	0	0	0	0	0
9	$\frac{1}{0!9!}$	$\frac{-2}{9!1!}$	$\frac{2^{2}}{9!2!}$	$\frac{-2^{3}}{9!3!}$	$\frac{2^{4}}{9!4!}$	$\frac{-2^{5}}{9!5!}$	$\frac{2^{6}}{9!6!}$	$\frac{-2^{7}}{9!7!}$	$\frac{1}{9!8!}$
10	0	0	0	0	0	0	0	0	0

Table 2: Coefficient Table for Example 1.

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)\,x^{k}t^{h} \qquad (10)$$

Using Table 3 we get following series solution

$$u(x,t)=\left[x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\frac{x^{9}}{9!}+\ldots..\right]+\left[-\frac{2x}{1!1!}+\frac{2x^{3}}{1!3!}-\frac{2x^{5}}{1!5!}+\ldots.\right]t$$
$$+\left[\frac{4x}{2!1!}-\frac{4x^{3}}{2!3!}+\frac{4x^{5}}{2!5!}+\ldots\ldots\ldots\ldots\right]t^{2}+\left[-\frac{8x}{3!1!}+\frac{8x^{3}}{3!3!}-\frac{x^{5}}{3!5!}+\ldots\ldots.\right]t^{3}$$
$$+\left[\frac{16x}{4!1!}-\frac{16x^{3}}{4!3!}+\frac{16x^{5}}{4!5!}+\ldots\ldots\ldots\ldots\right]t^{4}+\ldots\ldots\ldots\ldots\ldots$$

which is a series solution of (3) and converges to exact solution

$$u(x,t)=e^{-2t}\sin x$$

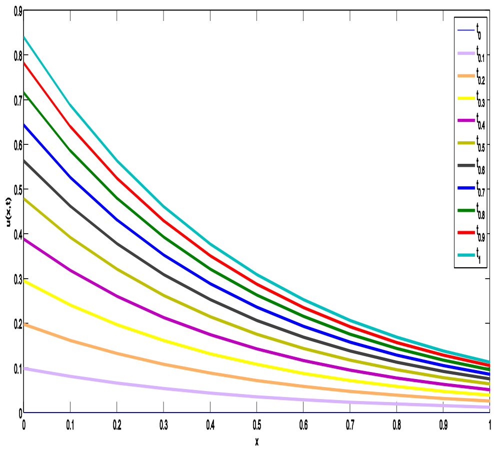


Figure 1: Graphical of u (x, t) w.r.t time.

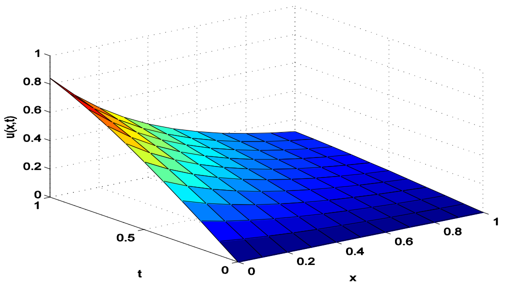


Fig. 1 :Three Dimensional graph of solution of u(x ,t) for linear homogenous diffusion equation.

Example 2: Consider the one dimensional heat transfer equation (diffusion equation) [5]

$$\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2}=0 \tag{11}$$

With initial condition

$$T(x,0)=g(x)=\sin 2\pi x \tag{12}$$

Applying DTM to (15) we get

$$(h+1)T(k,h+1)-\alpha(k+2)(k+1)T(k+2,h)=0 \tag{13}$$

From (16) we get

$$T(k,0)=\begin{cases} 0 & ,k=2n \\ \dfrac{(2\pi)^k}{k} & ,k=4n-3 \\ \dfrac{(-1)^k(2\pi)^k}{k!} & ,k=4n-1 \end{cases} \tag{14}$$

Using (13) & (14) we get following coefficient table

Table 3: Coefficient Table for Example 2.

k	h=0	h=1	h=2	h=3	h=4
0	0	0	0	0	0
1	$\frac{2\pi}{0!1!}$	$\frac{-\alpha(2\pi)^3}{1!1!}$	$\frac{\alpha^2(2\pi)^5}{1!2!}$	$-\frac{\alpha^3(2\pi)^7}{1!3!}$	$\frac{\alpha^4(2\pi)^9}{1!4!}$
2	0	0	0	0	0
3	$\frac{-(2\pi)^3}{0!3!}$	$\frac{\alpha(2\pi)^5}{3!1!}$	$\frac{-\alpha^2(2\pi)^7}{3!2!}$	$\frac{\alpha^3(2\pi)^9}{3!3!}$	$\frac{-\alpha^4(2\pi)^{11}}{3!4!}$
4	0	0	0	0	0
5	$\frac{(2\pi)^5}{5!0!}$	$\frac{-\alpha(2\pi)^7}{5!1!}$	$\frac{\alpha^2(2\pi)^9}{5!2!}$	$\frac{-\alpha^3(2\pi)^{11}}{5!3!}$	$\frac{\alpha^4(2\pi)^{13}}{5!4!}$
6	0	0	0	0	0
7	$\frac{-(2\pi)^7}{7!0!}$	$\frac{\alpha(2\pi)^9}{7!1!}$	$\frac{-\alpha^2(2\pi)^{11}}{7!2!}$	$\frac{\alpha^3(2\pi)^{13}}{7!3!}$	$\frac{-\alpha^4(2\pi)^{15}}{7!4!}$
8	0	0	0	0	0

$$T(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}T(k,h)\,x^k t^h$$

$$T(x,t)=\left[2\pi x-\frac{(2\pi x)^3}{3!}+\frac{(2\pi x)^5}{5!}-\frac{(2\pi x)^7}{7!}+\ldots\right]-\frac{\alpha(2\pi)^2t}{1!}\left[\frac{2\pi x}{1!}-\frac{(2\pi x)^3}{3!}+\frac{(2\pi x)^5}{5!}+\ldots\right]+\frac{\alpha^2(2\pi)^4t^2}{2!}\left[2\pi x-\frac{(2\pi x)^3}{3!}+\frac{(2\pi x)^5}{5!}+\ldots\right]-\frac{\alpha^3(2\pi)^8t^3}{1!}\left[\frac{2\pi x}{1!}-\frac{(2\pi x)^3}{3!}+\frac{(2\pi x)^5}{5!}+\ldots\right]\ldots$$

which the series solution of (11) and closed form of solution is given by

$$T(x,t)=e^{-4\pi^2\alpha t}.\sin 2\pi x.$$

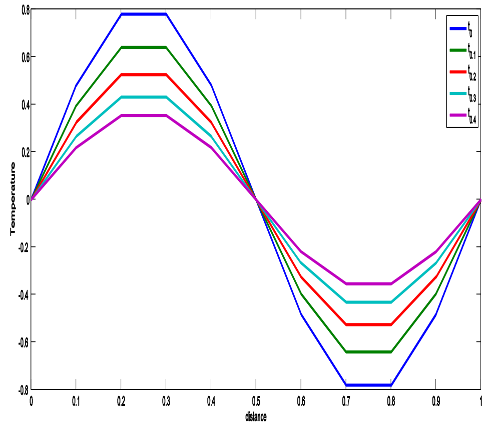


Fig.3: Solution of T(x,t) w.r.t to distance.

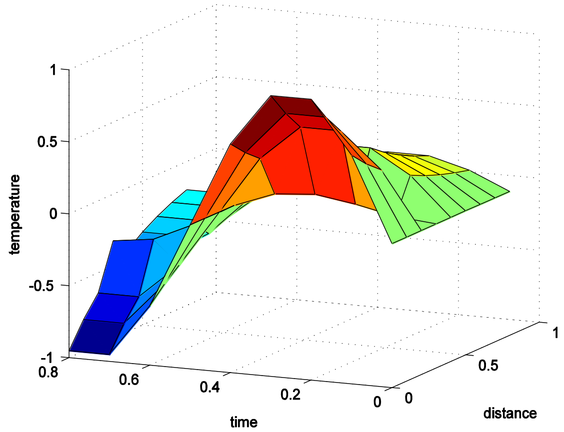


Fig.4: Solution of T(x, t) for different values of t and x at $\alpha=0.05$

Example 3: Consider the inhomogeneous diffusion equation [6]

$$u_t=u_{xx}+\cos x \tag{15}$$

with initial and boundary condition

$$u(x,0)=0 \tag{16}$$

$$u(0,t)=1-e^{-t} \tag{17}$$

$$u(\pi,t)=-1+e^{-t} \tag{18}$$

Applying DTM to (15) we get the following recursive formula and coefficient table

$$(h+1)U(k,h+1)=(k+2)(k+1)U(k+2,h)+\frac{1}{k!}\cos\left(\frac{\pi k}{2}\right) \tag{19}$$

On applying DTM on initial condition (16) we get

$$U(k,0)=0,k\geq 1. \tag{20}$$

From (17) we get

$$U(0,0)=0$$

$$U(0,1)=\frac{(-1)^{h-1}}{h!},\ h\geq 1 \tag{21}$$

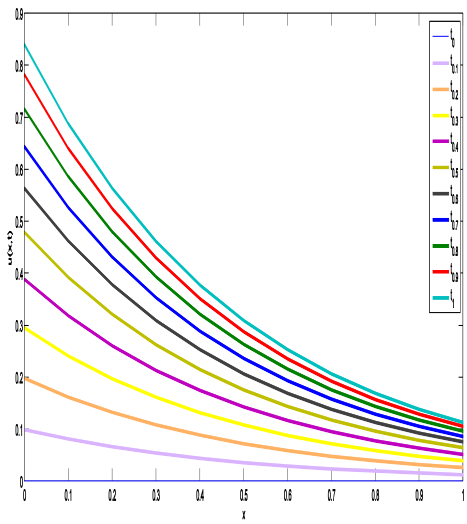
Using (19), (20) & (21) we get the following coefficient table.

Table 4: Coefficient Table for Example 3.

k	h=0	h=1	h=2	h=3	h=4	h=5	h=6	h=7	h=8
0	0	1	$\frac{-1}{2!}$	$\frac{1}{3!}$	$\frac{-1}{4!}$	$\frac{1}{5!}$	$\frac{-1}{6!}$	$\frac{1}{7!}$	$\frac{-1}{8!}$
1	0	0	0	0	0	0	0	0	0
2	0	$\frac{-1}{2!}$	0	$\frac{-2}{2!3!}$	$\frac{-4}{3!4!}$	$\frac{-20}{2!5!}$	0	0	0
3	0	0	0	0	0	0	0	0	0
4	0	$\frac{1}{4!}$	0	0	0	0	0	0	0
5	0	0	0	$\frac{-2^3}{5!3!}$	$\frac{2^4}{5!4!}$	$\frac{-2^5}{5!5!}$	$\frac{2^6}{5!6!}$	$\frac{-2^7}{5!7!}$	$\frac{2^8}{5!8!}$
6	0	$\frac{-1}{6!}$	0	0	0	0	0	0	0
7	0	0	0	$\frac{2^3}{7!3!}$	$\frac{-2^4}{7!4!}$	$\frac{2^5}{7!5!}$	$\frac{-2^6}{7!6!}$	$\frac{2^7}{7!7!}$	$\frac{-2^8}{7!8!}$
8	0	$\frac{1}{8!}$	0	0	0	0	0	0	0
9	0	0	0	$\frac{-2^3}{9!3!}$	$\frac{2^4}{9!4!}$	$\frac{-2^5}{9!5!}$	$\frac{2^6}{9!6!}$	$\frac{-2^7}{9!7!}$	$\frac{1}{9!8!}$
10	0	$\frac{-1}{10!}$	0	0	0	0	0	0	0

Using equation (10) and above coefficient table we get the exact solution

$$u(x,t) = \cos x(1-e^{-t})$$



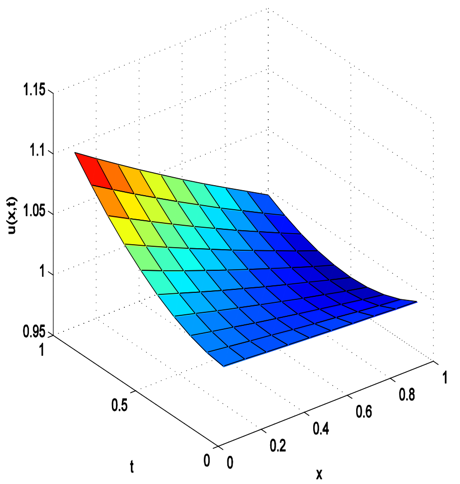


Fig.5: Solution of u (x, t) w.r.t time and distance.

IV. CONCLUSIONS

The present paper shows that DTM is one of the powerful method to solve the linear and non-linear diffusion equation and it can be applied to many other partial differential equations. The result obtain by it shows a good agreement with the existing result. DTM doesn't require to calculate the Adomain polynomial to find the solution as in ADM neither it requires linearization, discretization or perturbation as in VIM, HPM, it convert the given differential equation and initial and boundary condition into recursive formula it reduces the complicate computation and solution can be easily obtaining in the close form.

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