Uniqueness of χ^{c} - Partition of a graph

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Abstract A coloring of a graph G is called a complementary coloring of G if whenever two vertices u and v have distinct colors then u and v are adjacent. An n – coloring which is also complementary is called complementary n – coloring. Complementary chromatic number of a graph G is the largest integer k such that G admits a complementary k – coloring. This number is denoted as $\chi^{c}(G)$. Complementary coloring the vertices of G using maximum possible number of colors divides the vertex set V of G into disjoint subsets in such a way that vertices that are in distinct subsets are adjacent. Collection of these subsets is called χ^{c} - Partition of a graph G. In this paper we prove that any graph G admits unique χ^{c} – Partition.

Keywords: *Complementary Coloring, Complementary Chromatic number.*

1. *Introduction:*

We are very familiar with the concept of proper coloring of vertices of a graph G. It is a coloring in which two adjacent vertices must be assigned distinct colors. The minimum number of colors by which all the vertices of the graph G can be colored is called the Chromatic number of a graph. It is denoted by χ (G) or by χ. We know that proper coloring of vertices of a graph G using minimum number of colors partitions the vertex set V into minimum number of subsets that are independent. The collection of these subsets is called χ – Partition of G. In [2], we introduced the concept of Complementary Coloring of vertices of a graph. This coloring is actually improper coloring in general. That is, the adjacent vertices may receive same color. In this coloring, if two distinct vertices of a graph G are assigned distinct colors then they must be adjacent. The maximum number of colors that are required to color all the vertices of the graph G is called the Complementary Chromatic number of the graph. It is denoted by $\chi^{c}(G)$ or by χ^{c}. Complementary coloring the vertices of G using maximum possible number of colors divides the vertex set V of G into disjoint subsets in such a way that the vertices that are in distinct subsets are adjacent. Collection of these subsets is called χ^{c} - Partition of a graph G. Unlike proper coloring, this coloring yields color classes that may not be independent.

We begin with simple, finite, connected and undirected graph without isolated vertices. We first mention definitions.

2 *Definitions*

Definition 2.1 [1] An n – colouring of a graph G is a function f : V(G) → {1, 2,……….,n}(for some n≥ 1). This is called a proper n – colouring if whenever u and v are adjacent then f(u) ≠ f(v). A graph G is said to be n – colourable if it admits n – colouring and it is called a proper n – colourable graph if it admits a proper n – colouring.

Definition 2.2[1]: The chromatic number of a graph G is the smallest value of n such that G admits a proper n – colouring. This number is denoted as χ (G).

Definition 2.3[2]: A colouring of a graph G is called a complementary colouring of G if whenever two vertices u and v have distinct colours then u and v are adjacent. An n – colouring which is also complementary is called complementary n – colouring.

Definition 2.4[2]: Complementary chromatic number of a graph G is the largest integer k such that G admits a complementary k – colouring. This number is denoted as $\chi^{c}(G)$.

Definition 2.5[3]: Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ - Partition of the graph G.

Definition 2.6[4]: Complementary coloring of vertices of a graph G, by using maximum number of colors, yields maximum number of subsets (may not be independent) of vertex set of G called color classes of G. Such a partition of a vertex set of G is called a χ^{c} - Partition of the graph G.

3. Main Results:

Theorem 3. 1: Let $\Pi = \{V_1, V_2, \ldots, V_{\chi^c}\}$ be a χ^{c} - Partition of a graph G and $v \in V_i$ with $|V_i| \geq 2$, for some $i \in \{1, 2, 3, \ldots, \chi^c\}$.Then v is not adjacent to at least one $u \in V_i$. $(u \neq v)$

Proof: Suppose the theorem is not true. Then v is adjacent to every $u \in V_i$. $(u \neq v)$. Clearly $\Pi' = \{V_1, V_2, \ldots, V_{i-1}, V_i \setminus \{v\}, V_{i+1}, \ldots, V_{\chi^c}, \{v\}\}$ is χ^{c} – Partition of G, which is contradiction as the Complementary Chromaticity has increased. Hence the theorem.

Theorem 3.2: Any graph G = (V, E) admits unique χ^{c} – Partition.

Proof: Let $\Pi = \{V_1, V_2, ..., V_{\chi^c}\}$ be a χ^C - Partition of a graph G. Suppose the graph G admits another χ^C - Partition, say Π'. Then there exists u, $v \in V_i \in \Pi$ (for some i) such that they are in distinct color classes of Π', Say W_1 and W_2. Let $u \in W_1$ and $v \in W_2$. It is to be noted that u and v have to be adjacent. Note that $V_i \setminus \{u, v\} = S_1 \cup S_2 \cup S_3 \cup S_4$ where S_1 is the set of vertices that are neither adjacent to u nor to v both, S_2 is the set of vertices that are not adjacent to u but are adjacent to v, S_3 the set of vertices that are not adjacent to v but are adjacent to u and S_4 is the set of vertices that are adjacent to both u and v.

Claim: $S_1 = \phi$.

Suppose not. Then there exists a vertex, say w, in V_i, such that it is adjacent to neither u nor v. Hence u, v and w must be in same color class. So we get a contradiction. Hence we have $S_1 = \phi$.

Now the vertices in V_i, that are not adjacent to u, must be in W_1 and the vertices that are not adjacent to v, must be in W_2. Hence $S_2 \subset W_1$ and $S_3 \subset W_2$. Note that each vertex of S_2 have to be adjacent to each vertex of S_3. Here we note that $S_2 \cup \{u\} \subset W_1$ and $S_3 \cup \{v\} \subset W_2$.

Now let us discuss about the vertices in S_4.

Case I: $S_4 = \phi$.

In such case $V_i = S_2 \cup \{u\} \cup S_3 \cup \{v\}$. Note that each vertex in $S_2 \cup \{u\}$ is adjacent to each vertex in $S_3 \cup \{v\}$. Hence $\{V_1, V_2, ..., V_{i-1}, V_{i+1}, ..., V_{\chi^c}, S_2 \cup \{u\}, S_3 \cup \{v\}\}$ becomes a χ^C - Partition of a graph G, which is contradiction as the Complementary Chromatic number increases.

Case II: $S_4 \neq \phi$.

Clearly there does not exist any vertex x in S_4 such that it is not adjacent to some vertex y in W_1 and to some vertex z in W_2 for otherwise x, y and z have to be in the same color class which is not possible as y must be in W_1 and z must be in W_2. Hence any vertex in S_4 that is not adjacent to some vertex in W_1 (if any) must be adjacent to every vertex in W_2 and any vertex in S_4 that is not adjacent to some vertex in W_2 (if any) must be adjacent to every vertex of W_1. Now the collection of vertices, from S_4, Say X, that are not adjacent to some vertex of W_1 must be in W_1 and the collection of vertices, from S_4, Say Y, that are not adjacent to some vertex of W_2 must be in W_2.

Now the only vertices that are remaining in S_4 (if any) are the vertices that are adjacent to each vertex of both color classes W_1 and W_2.

Consider the collections $S_2 \cup \{u\} \cup X$ ($\subset V_i$) and $S_3 \cup \{v\} \cup Y$($\subset V_i$). Note that each vertex in color classes $V_1, V_2, ..., V_{i-1}, V_{i+1}, ..., V_{\chi^c}$ is adjacent to each vertex in both the sets $S_2 \cup \{u\} \cup X$ and $S_3 \cup \{v\} \cup Y$. Also each vertex in $S_4 \setminus \{X, Y\}$(if $S_4 \setminus \{X, Y\} \neq \phi$) is adjacent each vertex in $V_1, V_2, ..., V_{i-1}, V_{i+1}, ..., V_{\chi^c}$, $S_2 \cup \{u\} \cup X$ and $S_3 \cup \{v\} \cup Y$. Hence $V_1, V_2, ..., V_{i-1}, V_{i+1}, ..., V_{\chi^c}$, $S_2 \cup \{u\} \cup X$, $S_3 \cup \{v\} \cup Y$ are the color classes of some χ^C – Partition of G with zero (if $S_4 \setminus \{X, Y\} = \phi$), one or more color classes will be added by the vertices in S_4. Therefore we get a contradiction as the Complementary Chromatic number has increased.

Hence G admits unique χ^C - Partition.

We now mention few corollaries from the above theorem 3.2. But before that we require some notations. Hence we mention them first.

Notations: Let G = (V, E) be a graph. Then

(I) $G \setminus \{v\}$ is a sub graph of G obtained by removing a vertex 'v' from G.

(II) $G \setminus e$ is a sub graph of G obtained by removing an edge 'e' from G.

(III) Let u and v be two non adjacent vertices in G. Now add an edge 'e' between these two vertices. We obtain super graph of G denoted by G + e.

(IV) $V^i = \{v \in V \,/\, G \setminus \{v\}$ is disconnected$\}$

(V) $E^i = \{e \in E \,/\, G \setminus e$ is disconnected$\}$

Corollary 3.3: Let G = (V, E) be a graph with χ^C - Partition $\Pi = \{V_1, V_2, V_3, ..., V_{\chi^c}\}$, $v \in V_1$ (assuming without loss of generality), $v \notin V^i$ and the sub graph $G \setminus \{v\}$. Then $V_2, V_3, ..., V_{\chi^c}$ are color classes of $G \setminus \{v\}$ also.

Corollary 3.4: Let G = (V, E) be a graph with χ^C - Partition $\Pi = \{V_1, V_2, ..., V_{\chi^c}\}$ and the sub graph $G \setminus e$ where $e = \{u, v\} \in E$ and $e \notin E^i$.

(I) If $u, v \in V_1$ (assuming without loss of generality) then $V_2, V_3, ..., V_{\chi^c}$ are color classes of $G \setminus e$ also.

(II) If $u \in V_1$ and $v \in V_2$ (assuming without loss of generality) then $V_3, V_4 ....., V_{\chi^c}$ are color classes of $G \setminus e$ also.

Corollary 3.5: Let G = (V, E) be a graph with χ^C - Partition $\Pi = \{V_1, V_2, ..., V_{\chi^c}\}$ and u and v be two non adjacent vertices in G such that $u, v \in V_1$(assuming without loss of generality). Consider the super graph G + e obtained by adding an edge e between u and v. Then $V_2, V_3, ..., V_{\chi^c}$ are color classes of G + e also.

Theorem 3.6: Let G = (V, E) be a graph of order n. Then for any $v \in V$ then $\{v\}$ is a color class if and only if deg (v) = n – 1.

Proof: Obvious.

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