Additive-Quartic Functional Equations are Stable in Quasi-Banach Spaces

R.Kodandan[#1], R.Bhuvana Vijaya[*2]

[#1]*Assistant Professor in Science and Humanities Department ,*
Sreenivasa Institute of Technology and Management Studies,
Chittoor – 517127, Andhra Pradesh India.

#2 Department of Mathematics, Jawaharlal Nehru Technological University,
Anantapuramu - 515002, Andhra Pradesh India.

Abstract - *In this paper, the authors established the solution and generalized Ulam - Hyers stability of the additive-quartic functional equation*

$$f\left[2x_1+\sum_{i=2}^{n}x_i\right]+f\left[2x_1-\sum_{i=2}^{n}x_i\right]=$$
$$4\left[f\left(2x_1+\sum_{i=2}^{n}x_i\right)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)\right]$$
$$-3\left[f\left(\sum_{i=2}^{n}x_i\right)+f\left(-\sum_{i=2}^{n}x_i\right)\right]+10f(x_1)+14f(-x_1)$$

in Quasi Banach spaces .

Keywords — *additive-quartic mixed functional equation, Myers- Ulam stability, Quasi Banach spaces, p-Banach space.*

2010 Mathematics Subject Classification: 39B52, 32B72, 32B82.

I. Introduction

The study of perturbation problems for functional equations is related to a famous question of S.M. Ulam [28] concerning the stability of group homomorphisms. It was affirmatively answered by Hyers [12] for Banach spaces. It was further generalized and interesting results obtained by number of mathematicians ([2], [8], [22], [23], [25]). For more detailed information about such problems one can see ([2]-[5], [7], [9], [13]-[21]).

In this paper, the authors established the solution and generalized Ulam - Hyers stability of the additive-quartic functional equation of the form

$$f\left[2x_1+\sum_{i=2}^{n}x_i\right]+f\left[2x_1-\sum_{i=2}^{n}x_i\right]$$
$$=4\left[f\left(2x_1+\sum_{i=2}^{n}x_i\right)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)\right] \qquad (1.1)$$
$$-3\left[f\left(\sum_{i=2}^{n}x_i\right)+f\left(-\sum_{i=2}^{n}x_i\right)\right]+10f(x_1)+14f(-x_1)$$

in Quasi Banach spaces using direct method.

2. GENERAL SOLUTION OF (1.1)

In this section, we present the solution of the functional equation (1.1). Through out this section let X and Y be real vector spaces.

Theorem 2.1 *An odd function $f:X\to Y$ satisfies the functional equation (1.1) then f is additive.*

Proof. Let $f:X\to Y$ satisfies the functional equation (1.1). Letting $x_1,x_2,.......,x_n$ by $0,0,.......,0$ in (1.1), we get $f(0)=0$. Replacing $x_1,x_2,.......,x_n$ by $x,0,.......,0$ and $x,x,......,x$ in (1.1) respectively, and using oddness of f, we obtain $f(2x)=2f(x)$, $f(3x)=3f(x)$, for all $x\in X$. Replacing $x_1,x_2,.......,x_n$ by $x,y,0,.....,0$ and using oddness in (1.1), we get

$$f(2x+y)+f(2x-y)=4\left[f(x+y)+f(x-y)\right]+4f(x) \qquad (2.1)$$

Letting $x+y,x-y$ by u,v in (2.1), we obtain

$$f(x+u)+f(x+v)=4\left[f(u)+f(v)\right]-4f(x) \qquad (2.2)$$

Replacing u,v by y,y in (2.2), we obtain

$$2f(x+y)=8f(y)-4f(x) \qquad (2.3)$$

Intrechanging x and y, we get

$$2f(x+y)=8f(x)-4f(y) \qquad (2.4)$$

Adding (2.3) and (2.4), we obtain

$$f(x+y)=f(x)+f(y)$$

Hence the equation (1.1) is additive.

Lemma 2.2 *An even function* $f:X\to Y$ *satisfies the functional equation (1.1) then* f *is quartic.*

Proof. Let $f:X\to Y$ satisfies the functional equation (1.1). Using evenness of f and replacing $x_1,x_2,\dots\dots,x_n$ and $x,y,0,\dots\dots,0$, we get

$$f(2x+y)+f(2x-y) = 4\left[f(x+y)+f(x-y)\right]+12[f(x)+f(-x)] - 3[f(y)+f(-y)]-2[f(x)-f(-x)].$$

It is clear that f is quartic [16].

3. STABILITY RESULTS OF (1.1): DIRECT METHOD

Throughout this section, let us consider E_1 is a Quasi-Banach space with quasi-norm $\|.\|_{E_1}$ and E_2 is a p- Banach space with p -norm. $\|.\|_{E_2}$. Let K be the modulus of concavity of $\|.\|_{E_2}$. Define a mapping $f:E_1\to E_2$ by

$$\begin{aligned} Df\left(x_1,x_2,\dots\dots,x_n\right) &= f\left[2x_1+\sum_{i=2}^{n}x_i\right]+f\left[2x_1-\sum_{i=2}^{n}x_i\right] \\ &-4\left[f\left(2x_1+\sum_{i=2}^{n}x_i\right)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)\right] \\ &+3\left[f\left(\sum_{i=2}^{n}x_i\right)+f\left(-\sum_{i=2}^{n}x_i\right)\right]-10f(x_1)-14f(-x_1) \end{aligned} \qquad (3.1)$$

for all $x_i\in E_1$, $i=1,2,\dots.,n$ and we state the following Lemma 3.1 [15] without proof, it will be useful in proving our theorems.

Lemma 3.1

Let $0<p\le 1$ *and let* $x_1,x_2,\dots\dots\dots,x_n$ *be non negative real numbers then*

$$\left(\sum_{i=1}^{n}x_i\right)^p \le \left(\sum_{i=1}^{n}x_i^p\right). \qquad (3.2)$$

Theorem 3.2

Let $\phi:\underbrace{E_1\times E_1\times,\dots\dots\dots\times E_1}_{n\ times}\to[0,\infty)$ *be a function such that for all* $x_i\in E_1$, $i=1,2,\dots.,n$

$$\lim_{n\to\infty}(16)^n\phi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},\dots.,\frac{x_n}{2^n}\right)=0 \qquad (3.3)$$

And

$$\sum_{i=1}^{\infty}(16)^{ip}\phi^p\left(\frac{x_1}{2^i},\frac{x_2}{2^i},\dots.,\frac{x_n}{2^i}\right)<\infty \qquad (3.4)$$

for all $x_i\in E_1$ and for all $x_1,x_2\in\{0,x\}$, $x_i\in\ 0$, where $i=3,4,\dots.,n$

Suppose that an even function $f:E_1\to E_2$ with $f(0)=0$ satisfies the inequality

$$\|Df(x_1,x_2,\dots\dots,x_n)\|_{E_2}\le\phi(x_1,x_2,\dots\dots,x_n)\quad \forall x_i\in E_1 \qquad (3.5)$$

Then the limit

$$Q(x)=\lim_{n\to\infty}16^n f\left(\frac{x}{2^n}\right) \qquad (3.6)$$

exists for all $x\in E_1$ and $Q:E_1\to E_2$ is a unique quartic function satisfying

$$\|f(x)-Q(x)\|_{E_2}\le\frac{k}{16}\,\psi_e(x)^{\frac{1}{p}},\quad\forall x\in E_1 \qquad (3.7)$$

where

$$\psi_e(x)=\sum_{i=1}^{\infty}\frac{(16)^{ip}}{2^p}\left\{\phi^p\left(\frac{x}{2^i},0,0,\dots,0\right)+\phi^p\left(0,\frac{x}{2^i},0,\dots,0\right)\right\}$$

for all $x\in E_1$.

Proof.

Using evenness of f and replacing $x_1,x_2,\dots\dots,x_n$ and $x,y,0,\dots\dots,0$ in (3.5), we get

$$\left\|\begin{matrix} f(2x+y)+f(2x-y)- \\ 4\left[f(x+y)+f(x-y)\right] \\ -24f(x)+6f(y) \end{matrix}\right\|_{E_2}\le\phi\left(x,y,0,\dots\dots,0\right) \qquad (3.8)$$

for all $x,y\in E_1$ Replacing x,y by y,x in (3.8) and using evenness, we have obtain

(3.2)

$$\left\| \begin{matrix} f(x+2y)+f(x-2y)- \\ 4\ f(x+y)+f(x-y) \\ -24f(y)+6f(x) \end{matrix} \right\|_{E_2} \le \phi\ y,x,0,.....,0 \qquad (3.9)$$

for all $x,y \in E_1$,from (3.8) and (3.9) and replacing y by 0, we have

$$\|f(2x)-16f(x)\|_{E_2} \le \frac{k}{2}\begin{bmatrix} \phi\ x,0,0,.....,0 \\ +\phi\ 0,x,0,.....,0 \end{bmatrix}, \ \forall x \in E_1. \qquad (3.10)$$

which can be written as

$$\|f(2x)-16f(x)\|_{E_2} \le \frac{k}{2}\psi_e(x) \ , \ \forall x \in E_1,$$

and

$$\psi(x)=\frac{1}{2}\begin{bmatrix} \phi\ 0,x,0,.....,0 \\ +\phi\ 0,x,0,.....,0 \end{bmatrix}, \ \forall x \in E_1, \qquad (3.11)$$

in equation (3.10), replace x by $\frac{x}{2^{n+1}}$ and multiplying both sides by $(16)^n$, we have

$$\left\|(16)^{n+1} f\left(\frac{x}{2^{n+1}}\right)-(16)^n f\left(\frac{x}{2^n}\right)\right\|_{E_2} \le k(16)^n \psi_e\left(\frac{x}{2^{n+1}}\right), \quad \forall x \in E_1, \qquad (3.12)$$

for all non-negative integers n , since $x \in E_2$ is a p-Banach space and using (3.12) , we obtain

$$\left\|(16)^{n+1} f\left(\frac{x}{2^{n+1}}\right)-(16)^n f\left(\frac{x}{2^n}\right)\right\|_{E_2}^p \le \sum_{i=m}^{n}\left\|(16)^{i+1} f\left(\frac{x}{2^{i+1}}\right)-(16)^i f\left(\frac{x}{2^i}\right)\right\|_{E_2}^p \le k^p\sum_{i=m}^{n}(16)^{ip}\psi^p\left(\frac{x}{2^{i+1}}\right) \qquad (3.13)$$

for all non-negative integers n and m with $n \ge m$ and all $x \in E_1$. Now $0 < p \le 1$ and with the help of Lemma 3.1, the equation (3.11) can be written as

$$\psi^p\ x = \frac{1}{2^p}\begin{bmatrix} \phi^p\ x,0,0,.....,0 \\ +\phi^p\ 0,x,0,.....,0 \end{bmatrix}, \ \forall x \in E_1. \ (3.14)$$

Therefore it follows from (3.4) and (3.14) that

$$\sum_{i=1}^{\infty}(16)^{ip}\psi^p\left(\frac{x}{2^i}\right) \le \infty \qquad (3.15)$$

$x \in E_1$. Therefore, we conclude from (3.13) and (3.15) that the sequence $\left\{(16)^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in E_1$, since E_2 is complete, the sequence $\left\{(16)^n f\left(\frac{x}{2^n}\right)\right\}$ converges for all $x \in E_1$. Now we define the mapping by $Q: E_1 \to E_2$ by (3.10) (3.6) for all $x \in E_1$. Allowing $n \to \infty$ in (3.13) , we get

$$\|f\ x - Q\ x\|_{E_2}^p \le k^p\sum_{i=0}^{\infty}(16)^{ip}\psi^p\left(\frac{x}{2^{i+1}}\right) = \frac{k^p}{(16)^p}\sum_{i=0}^{\infty}\psi^p\left(\frac{x}{2^i}\right), \ \forall x \in E_1. \qquad (3.16)$$

Use (3.11) in the equation (3.16), we arrive the result (3.7). Now, we show that Q is a quartic it follows from (3.3),(3.5) and (3.6),

$$\|DQ(x_1,x_2,...,x_n)\|_{E_2} = \lim_{n\to\infty}(16)^n\left\|Df\left(\frac{x_1}{2^n},\frac{x_2}{2^n},...,\frac{x_n}{2^n}\right)\right\|_{E_2} \le (16)^n\phi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},...,\frac{x_n}{2^n}\right), \quad \forall x_1,x_2,..,x_n \in E_1.$$

Therefore the mapping $Q: E_1 \to E_2$ satisfies (1.5). Since $Q(x)=0$, then by Lemma 2.1, we obtain that the mapping $Q: E_1 \to E_2$ is quartic. To prove the uniqueness of Q, let $Q': E_1 \to E_2$ be another quartic mapping satisfying (3.7). Since

$$\lim_{n\to\infty}(16)^n\sum_{i=1}^{\infty}(16)^{ip}\phi^p\left(\frac{x_1}{2^{n+i}},\frac{x_2}{2^{n+i}},...,\frac{x_n}{2^{n+i}}\right) = \lim_{n\to\infty}(16)^{ip}\phi^p\left(\frac{x_1}{2^i},\frac{x_2}{2^i},...,\frac{x_n}{2^i}\right)=0, \quad \forall x_1,x_2,..,x_n \in E_1,$$

and for all $x_1, x_2 \in \{0, x\}, x_i \in \{0\}$ where $i = 3,4,....,n$ then

$$\lim_{n\to\infty}(16)^{np}\psi_e\left(\frac{x}{2^n}\right)=0, \quad \forall x\in E_1. \qquad (3.17)$$

It follows from (3.7) and (3.17),

$$\left\|Q(x)-T(x)\right\|_{E_2}^p = \lim_{n\to\infty}(16)^n\left\|f\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right\|_{E_2}^p$$

$$\le \lim_{n\to\infty}(16)^{np}\psi_e\left(\frac{x}{2^n}\right)=0 \ , \quad \forall x\in E_1 \ ,$$

so $Q = T$.

Theorem 3.3

Let $\phi : E_1 \times E_1 \times ,........\times E_1 \to [0,\infty)$ be a function such that for all $x_i \in E_1$.

$$\lim_{n\to\infty}\frac{1}{(16)^n}\phi\left(2^n x_1, 2^n x_2, ..., 2^n x_n\right)=0,$$

$$\forall x_1,....x_n \in E_1 \qquad (3.18)$$

And

$$\sum_{i=1}^{\infty}\frac{1}{(16)^{ip}}\phi^p\left(2^i x_1, 2^i x_2, ..., 2^i x_n\right)<\infty \qquad (3.19)$$

for all $x_1, x_2 \in \{0, x\}, x_i \in \{0\}$, where $i = 3,4,....,n$. Suppose that an even function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality (3.5) for all $x_i \in E_1$. Then the limit

$$Q(x)=\lim_{n\to\infty}\frac{1}{(16)^n}f\left(2^n x\right), \qquad (3.20)$$

exists for all $x \in E_1$ and $Q : E_1 \to E_2$ is a unique quartic function satisfying

$$\| f(x)-Q(x)\|_{E_2}$$

$$\le \frac{k}{16}\ \psi_e(x)^{\frac{1}{p}}, \forall x\in E_1, \qquad (3.21)$$

where

$$\psi_e = \sum_{i=1}^{\infty}\frac{1}{2^p(16)^{ip}}\left\{\begin{array}{l}\phi^p(2^i x,0,0,...,0)\\ \quad +\phi^p(0,2^i x,0,...,0)\end{array}\right\}$$

for all $x \in E_1$.

Proof.

If we replacing x by $2^n x$ in (3.10) and dividing by $(16)^{n+1}$ on both sides of (3.10), we obtain

(3.17)

$$\left\|\frac{1}{(16)^{n+1}}f\left(2^{n+1}x\right)-\frac{1}{(16)^n}f\left(2^n x\right)\right\|_{E_2}$$

$$\le \frac{k}{(16)^{n+1}}\psi\left(2^n x\right) \qquad (3.22)$$

for all $x \in E_1$ and for all non-negative integers n . Since E_2 is a p-Banach space, using (3.18), we obtain

$$\left\|\frac{1}{(16)^{n+1}}f\left(2^{n+1}x\right)-\frac{1}{(16)^m}f\left(2^m x\right)\right\|_{E_2}^p$$

$$\le \sum_{i=m}^{n}\left\|\frac{1}{(16)^{i+1}}f\left(2^{i+1}x\right)-\frac{1}{(16)^i}f\left(2^i x\right)\right\|_{E_2}^p$$

$$\le \frac{k^p}{(16)^p}\sum_{i=m}^{n}\frac{1}{(16)^{ip}}\psi^p\left(2^{i+1}x\right), \qquad (3.23)$$

for all non-negative integers n and m with $n \ge m$ and all $x \in E_1$. Since (3.19)

$$\sum_{i=0}^{\infty}\frac{1}{(16)^{ip}}\psi^p\left(2^i x\right)<\infty, \quad \forall x\in E_1 \ ,$$

then (3.23) implies that the sequence $\left\{\frac{1}{(16)^n}f\left(2^n x\right)\right\}$ is a Cauchy sequence for all $x \in E_1$, since E_2 is complete, the sequence $\left\{\frac{1}{(16)^n}f\left(2^n x\right)\right\}$ converges for all $x \in E_1$. Now we define the mapping by $Q : E_1 \to E_2$ (3.20) by for all $x \in E_1$. Letting $m = 0$ and $n \to \infty$ in (3.23), we get

$$\| f(x)-Q(x)\|_{E_2}^p$$

$$\le \frac{k^p}{(16)^p}\sum_{i=0}^{\infty}\frac{1}{(16)^{ip}}\psi^p\left(2^i x\right), \qquad (3.24)$$

$$\forall x\in E_1 .$$

Use (3.11) in the equation (3.24) , we arrive the result (3.21). Now using (3.24), (3.22) in the equation (3.4), we obtain

$$\|DQ(x_1,x_2,\ldots,x_n)\|_{E_2} \le \lim_{n\to\infty}\frac{1}{(16)^n}\phi\left(2^n x_1,2^n x_2,\ldots,2^n x_n\right), \quad \forall x_1,..,x_n\in E_1.$$

Therefore the mapping $Q:E_1\to E_2$ satisfies (1.5). Since $Q(x)=0$ then by Lemma 2.1, we obtain that the mapping Q is Quartic. Uniqueness is proved in similar manner, as in the proof of Theorem 3.2.

Corollary 3.4. Let λ, r be non negative real numbers such that $r<4$, suppose that an even function $f:E_1\to E_2$ which satisfies the inequality

$$\|D f(x_1,x_2,\ldots,x_n)\|_{E_2}\le \lambda\left[\sum_{i=1}^{n}\|x_i\|_{E_1}^{r}\right], \ \forall x_i\in E_1.$$

Then there exists a unique quartic function $Q:E_1\to E_2$ satisfies

$$\|f(x)-Q(x)\|_{E_2}\le\frac{k}{32}\left[\frac{1}{\left|1-2^{(r-4)p}\right|}\|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \ \forall x\in E_1$$

The proof of the following theorem is similar to that of Theorem 3.1 for f is even. Hence the details of the proof are omitted.

Theorem 3.5

Let $\phi:E_1\times E_1\times,\ldots\ldots\times E_1\to[0,\infty)$ be a function such that for all $x_i\in E_1$ where $i=1,2,\ldots,n$

$$\lim_{n\to\infty}2^n\phi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},\ldots,\frac{x_n}{2^n}\right)=0$$

and

$$\sum_{i=1}^{\infty}2^{ip}\phi\left(\frac{x_1}{2^i},\frac{x_2}{2^i},\ldots,\frac{x_n}{2^i}\right)<\infty, \qquad \forall x_1,x_2,\ldots,x_n\in E_1$$

and for all $x_1,x_2\in\{0,x\}$ and $x_i\in\{0\}$ where $i=3,4,\ldots,n$

Suppose that an odd function $f:E_1\to E_2$ satisfies the inequality (3.5).

Then the limit

$$A(x)=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x\in E_1$ and $A:E_1\to E_2$ is a unique additive function satisfying

$$\|f(x)-A(x)\|_{E_2}\le\frac{k}{2}\left(\varphi_o(x)\right)^{\frac{1}{p}} \qquad \forall x\in E_1$$

where

$$\varphi_o(x)=\sum_{i=1}^{\infty}\left(2^{i-1}\right)^p\left\{\phi\left(\frac{x}{2^i},0,0,\ldots,0\right)+\phi\left(0,\frac{x}{2^i},0,\ldots,0\right)\right\}$$

for all $x\in E_1$.

Proof.

The proof of the following theorem is similar to that of Theorem 3.1 for f is odd. Hence the details of the proof are omitted.

Theorem 3.6

Let $\phi:E_1\times E_1\times,\ldots\ldots\times E_1\to[0,\infty)$ be a function such that

$$\lim_{n\to\infty}\frac{1}{2^n}\phi\left(2^n x_1,2^n x_2,\ldots,2^n x_n\right)=0, \ \forall x_1,x_2,\ldots,x_n\in E_1,$$

and $\sum_{i=1}^{\infty}\frac{1}{2^{ip}}\phi^p\left(2^i x_1,2^i x_2,\ldots,2^i x_n\right)<\infty$

for all $x_1,x_2\in\{0,x\}, x_i\in\{0\}$, where $i=3,4,\ldots,n$. Suppose that an odd function $f:E_1\to E_2$ with $f(0)=0$ satisfies the inequality (3.5)

Then the limit $A(x)=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right)$ exists for all $x\in E_1$ and $A:E_1\to E_2$ is a unique additive function satisfying

$$\|f(x)-A(x)\|_{E_2}\le\frac{k}{2}\left(\varphi_o(x)\right)^{\frac{1}{p}} \qquad \forall x\in E_1$$

where

$$\varphi_o=\sum_{i=1}^{\infty}\frac{1}{\left(2^{i+1}\right)^p}\left\{\phi^p(2^i x,0,0,\ldots,0)+\phi^p(0,2^i x,0,\ldots,0)\right\}$$

for all $x\in E_1$.

Proof.

The proof of the following theorem is similar to that of Theorem 3.2 for f is odd

Corollary 3.7. Let λ be non negative real number and be r real number such that $r<1$, suppose that an odd function $f:E_1\to E_2$ which satisfies the inequality (3.21) Then there exists a unique additive function $A:E_1\to E_2$ satisfies

$$\| f(x)-A(x)\|_{E_2} \le \frac{k}{4}\left[\frac{1}{\left|1-2^{(4-r)p}\right|}\|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \quad \forall x\in E_1$$

The proof of the following theorem is similar to that of Theorem 3.7 for f is odd. Hence the details of the proof are omitted.

Theorem 3.8.

Let $\phi: E_1\times E_1\times,........\times E_1 \to[0,\infty)$ be a function satisfies (3.3) and (3.4) for all $x_i\in E_1$,
and for all $x_1,x_2\in\{0,x\}$, $x_i\in\{0\}$ where $i=3,4,....,n$. Suppose that a function $f:E_1\to E_2$
satisfies the inequality (3.5) with $f(0)=0$ for all $x\in E_1$, then there exists a unique quartic function $Q:E_1\to E_2$ and a unique additive function $A:E_1\to E_2$ satisfies (1.5) and

$$\| f(x)-Q(x)-A(x)\|_{E_2} \le \frac{k^3}{32}\left\{\begin{array}{l}\psi_e(x)+\psi_e(-x)^{\frac{1}{p}}\\ +8\ \varphi_o(x)+\varphi_o(-x)^{\frac{1}{p}}\end{array}\right\}, \forall x\in E_1$$

Where $\psi_e(x)$ and $\varphi_o(x)$ are already given.

Proof.

The proof of this Theorem follows from Theorem 3.1 and Theorem 3.5 and so the proof is omitted here.

Theorem 3.9.

Let $\phi: E_1\times E_1\times,........\times E_1 \to[0,\infty)$ be a function satisfies (3.36) and (3.37) for all $x_i\in E_1$,
and for all $x_1,x_2\in\{0,x\}$, $x_i\in\{0\}$ where $i=3,4,....,n$. Suppose that a function $f:E_1\to E_2$
satisfies the inequality (3.5) with $f(0)=0$ for all $x\in E_1$, then there exists a unique quartic function $Q:E_1\to E_2$ and a unique additive function $A:E_1\to E_2$ satisfies (1.5) and

$$\| f(x)-Q(x)-A(x)\|_{E_2} \le \frac{k^3}{32}\left\{\begin{array}{l}\psi_e(x)+\psi_e(-x)^{\frac{1}{p}}\\ +8\ \varphi_o(x)+\varphi_o(-x)^{\frac{1}{p}}\end{array}\right\}, \quad \forall x\in E_1$$

Where $\psi_e(x)$ and $\varphi_o(x)$ are already defined , for all $x\in E_1$.

Proof.

The proof of this Theorem follows from Theorem 3.3 and Theorem 3.7 and it is very similar to the Theorem 3.8 and so the proof is omitted here.

Corollary 3.10. Let λ be non negative real number and be r real number such that $r<4$, suppose that an function $f:E_1\to E_2$ which satisfies the inequality (3.21) then there exists a unique quartic function $Q:E_1\to E_2$ and a unique additive function $A:E_1\to E_2$ satisfies (1.5) then

$$\| f(x)-Q(x)-A(x)\|_{E_2} \le \frac{k}{2^{p+1}}\left[\left\{\begin{array}{l}\frac{1}{8\left|1-2^{(r-4)p}\right|}\\ +\frac{1}{\left|1-2^{(r-1)p}\right|}\end{array}\right\}\|x\|_{E_1}^{rp}\right]^{\frac{1}{p}}, \quad \forall x\in E_1 \tag{3.51}$$

REFERENCES

[1] J.Aczel and Dhombres J., Functional Equations in Several Variables, Cambridge Univ, Press, 1989.

[2] T.Aoki , On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64-66.

[3] M.Arunkumar.,J.M.Rassias., On the generalized Ulam-Hyers stability of an AQ-mixed type functional equation with counter examples, Far East Journal of Applied Mathematics, Volume 71, No. 2, (2012), 279-305.

[4] M. Arunkumar.,P.Agilan., Additive Quadratic functional equation are stable in Banach space: A Fixed Point Approach, International Journal of pure and Applied Mathematics, Vol. 86 No.6, 951-963, (2013).

[5] M.Arunkumar., P.Agilan., Additive Quadratic functional equation are stable in Banach space: A Direct Method, Far

East Journal of Mathematical Sciences, Volume 80, No. 1, (2013), 105 – 121.

[6] S. S. Chang, Y. J. Cho, and S. M. Kang, Nonlinear Operator Theory in Probabilistic Metric Spaces, Nova Science Publishers, Huntington, NY, USA, 2001.

[7] Czerwik S., Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.

[8] P.Gavruta ., A generalizationof the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184 (1994), 431-436.

[9] M. Eshaghi Gordji., N.Ghobadipour .,J.M. Rassias ., Fuzzy stability of additive quadratic functionalEquations, arXiv:0903.0842v1 [math.FA] 4 Mar 2009.

[10] O.Hadzic ., Fixed Point Theory in Probabilistic Metric Spaces, vol. 536 of Mathematics and its Applications, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[11] O.Pap E Pap . and Budincevic M., Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces, Kybernetika, vol. 38, no. 3, (2002) 363-382.

[12] D.H.Hyers., On the stability of the linear functional equation, Proc. Nat. Acad.Sci.,U.S.A., 27, (1941), 222-224.

[13] D.H.Hyers .,G. Isac., Th.M.Rassias ., Stability of unctional equations in several variables, Birkhauser, Basel, 1998.

[14] H.M..Jun.. M.Kim., On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation, Bull. Korean Math. Soc. 42(1), (2005), 133-148.

[15] S.M.Jung., Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.

[16] S.H.Lee., S.M.Im ., I.S.Hwang., Quartic functional equations, J. Math. Anal. Appl., 307, (2005), 387-394.

[17] P.L.Kannappanl., Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, 2009.

[18] S.Murthy.,M.Arunkumar., G.Ganapathy., P. Rajarethinam ., Stability of mixed type additive quadratic functional equation in Random Normed space, International Journal of Applied Mathematics Vol. 26. No. 2 (2013), 123-136.

[19] A.Najati., M.B.Moghimi ., On the Stability of a quadratic and additive functional equation, J. Math. Anal. Appl. 337 (2008), 399-415.

[20] C.Park., Orthogonal Stability of an Additive-Quadratic Functional Equation, Fixed Point Theory and Applications 2001 2011:66.

[21] J.M.Rassias., M Arunkumar ., S.Ramamoorthi., Stability of the Leibniz additive-quadratic functional equation in Quasi-Beta normed space: Direct and fixed point methods, Journal of Concrete and Applicable Mathematics (JCAAM), (Accepted).

[22] J.M.Rassias., On approximately of approximately linear mappings by linear mappings, J. Funct. Anal. USA, 46, (1982) 126-130.

[23] Th.M.Rassias., On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72 (1978), 297-300.

[24] Th.M.,Rassias., Functional Equations, Inequalities and Applications, Kluwer Acedamic Publishers, Dordrecht, Bostan London, 2003.

[25] K.Ravi., M.Arunkumar. and Rassias J. M., On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[26] B. schweizer and A. Sklar, Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing, New York, NY, USA, 1983.

[27] A.N.Sherstnev ., On the notion of a random normed space, Doklady Akademii Nauk SSSR, vol. 149, (1963) 280-283, (Russian).

[28] S.M.Ulam., Problems in Modern Mathematics, Science Editions, Wiley, New York, 1964.

[29] G.Zamani Eskandani., Hamid Vaezi, Y.N.Dehghan ., Stability of A Mixed Additive and Quadratic Functional Equation In Non-Archimedean Banach Modules, Taiwanese Journal of Mathematics, Vol. 14, No. 4, (2010), 1309-1324.