NEW GENERALIZATION OF FRACTIONAL KINETIC EQUATION USING MULTIVARIABLE ALEPH-FUNCTION

F.Y. AYANT[1]

1 Teacher in High School , France

Abstract : Recently A.Choudhary et al [5] use the multivariable H-function for solving generalized fractional kinetic equation. Motivated by the recent work , we present a new generalization of fractional kinetic equation by using multivariable Aleph-function. The new generalization can be used for the computation of the change of chemical composition in stars like the sun. The solution of the generalized fractional kinetic equation involving multivariable Aleph-function is obtained with help of the Laplace transform method. Further, the same generalized fractional kinetic equation is solved by using the Sumudu transform method. The solution of the proposed problem is presented in a compact form in term of the multivariable Aleph-function. Some special cases, involving the multivariable H-function, the H-function of two variables and the Aleph function of one variable are also considered.

Keywords : Multivariable Aleph-function. Fractional Kinetic equation. Laplace transform. Sumudu Transform. Riemann-Liouville fractional integral.

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1. INTRODUCTION

The fractional calculus has many important developments and concepts in mathematics initiates with fractional kinetic models (kinetic equation). The great use of Mathematical Physics in imposing astrophysical problems have pulled stargazers and physicists to pay more attention in available mathematical tools that can be used to solve several problems of astrophysics. The importance of fractional kinetic equation has been increased by virtue of its occurrence in certain problem related to kinetic motion of particles in science and engineering. The thermal and hydrostatics equilibrium are pretended as spherically symmetric, non-magnetic, non-rotating, self gravitating model of a star like sun. The properties of star arecharacterized by its mass, brightness effective surface temperature, radius, central density, temperature etc. Turn over an arbitrary reaction characterized by N = N (t) which is dependent on time. It is possible to compute rate of change dN/dt to a balance between the demolition rate d and the production rate p of N , that is
dN/dt = −d + p. In general, through interaction mechanism, demolition and production depend on the quantity N itself :
d = d (N) or p = p (N). This dependence is complicated for the demolition of production at time depends not only on N(t), but also on the proceding history N (ī) , ī < t , of the variable N.
This may be formally represented by [6].

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{1.1}$$

Where N_t denote the function defined by N_t (ṫ) = N (t – ṫ) , ṫ > 0.
Haubold and Mathai [6] studied a special case of this equation , when instance of changes in quantity N (t) are unvalued , is given by the equation :

$$\frac{dN_i}{dt} = -c_i N_i(t) \tag{1.2}$$

with the initial condition that $N_t(t=0) = N_0$ is the number density of species i at time t = 0; constant $c_i > 0$, known as standard kinetic equation. The solution of the Eq. (2) is give :

$$N_i(t) = N_0 exp(-c_i t) \tag{1.3}$$

Alternative form of Eq. (2) can be obtained on integration :

$$N(t) - N_0 = c_0 D_t^{-1} N(t) \tag{1.4}$$

where ${}_0D_t^{-1}$ is the standard integral operator. Haubold and Mathai [6] have given the fractional generalization of the standard kinetic Eq, (2) as

$$N(t) - N_0 = c_0 D_t^{-\upsilon} N(t) \tag{1.5}$$

Where ${}_0D_t^{-\upsilon}$ is the well known Riemann-Liouville fractional integral operator (Oldhman and Spanier [10] ; Samko et al [11] ; Miller and Ross [9] , Srivastava and Saxena [14]) defined by :

$${}_0D_t^{-\upsilon} = \frac{1}{\Gamma(\upsilon)} \int_0^t (t-u)^{\upsilon-1} f(u)\,\mathrm{d}u \,, \qquad \mathrm{Re}\,(\,\upsilon\,) > 0 \tag{1.6}$$

The solution of the fractional Kinetic Eq. (6) is given by Haubold and Mathai [6] as:

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\upsilon n+1)} (ct)^{n\upsilon} \tag{1.7}$$

2. MATHEMATICAL PREREQUISITES

The multivariable aleph-function is defined in term of multiple Mellin-Barnes type integral :

$$\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c|c} z_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & \end{array} \right.$$

$$\begin{array}{l} [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}] \;\; ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]: \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]: \end{array}$$

$$\left. \begin{array}{l} [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;\;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \phi_k(s_k) z_k^{s_k}\, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{2.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i \prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{2.2}$$

and $$\phi_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{2.3}$$

where $j = 1$ to r and $k = 1$ to r

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{2.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)} \tag{2.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1} \ldots |z_r|^{\alpha_r}) , max(|z_1| \ldots |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1} \ldots |z_r|^{\beta_r}) , min(|z_1| \ldots |z_r|) \rightarrow \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{2.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{2.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{2.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{2.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{2.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{2.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} z_1 & \mathrm{A : C} \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ \cdot & \\ z_r & \mathrm{B : D} \end{array}\right) \tag{2.12}$$

A new integral transform introduced by Watugala [15] , so-called the Sumudu transform. For further details and fundamental properties of the Sumudu transform , see [14 , 1 , 2 ,4]. Let A be the class of exponentially bounded function

$f : \mathbb{R} \rightarrow \mathbb{R}$, that is : $| f(t)| < Me^{-t/\tau_1}$ if $,t \leqslant 0$, else $| f(t)| > Me^{t/\tau_2}, t \geqslant 0$

where M, τ_1 and τ_2 are positive real constants,

the Sumudu transform is defined by :

$$\mathrm{G(u) = S[f(t)]} = \int_0^{\infty} f(ut)e^{-t}\, \mathrm{d}t\, , u \in (-\tau_1, \tau_2) \tag{2.13}$$

The Sumudu transform has been shown to be the theoretical dual of the Laplace transform.

The Riemann-Liouville fractional integral order υ is defined by [11 ,14],

$$_0D_t^{-\upsilon} N(x,t) = \frac{1}{\Gamma(\upsilon)} \int_0^t (t-u)^{\upsilon-1} N(x,u)\, \mathrm{d}u, Re(\upsilon) > 0 \tag{2.14}$$

The Sumudu transform of Riemann-Liouville fractional integral order υ is defined as [8]:

$$S\left\{_0D_t^{-\upsilon} f(t); u\right\} = u^{\upsilon} \overline{f}(\mathrm{\ u\ }) \tag{2.15}$$

3. Generalized Fractional Kinetic Equation

Lemma 3.1 The Laplace transform of the multivariable Aleph-function as follows

$$L\{t^{\lambda-1}\aleph(z_1t^{\mu},\cdots,z_rt^{\mu})\}=u^{-\lambda}\aleph^{0,\mathfrak{n}+1:V}_{U_{10}:W}\left(\begin{array}{c|c}\frac{z_1}{u} & (1-\lambda;\mu,\cdots,\mu),A:C\\ \cdot & \\ \cdot & \\ \cdot & \mathrm{B:D}\\ \frac{z_r}{u} & \end{array}\right) \tag{3.1}$$

where $\lambda,z_1,\cdots,z_r,u\in\mathbb{C}$, $Re(u)>0,\mu>0$, $U_{10}=p_i+1,q_i,\tau_i;R$

Proof For convenience , we denote the left side of (3.1) by Ł.

$$Ł=\frac{1}{(2\pi\omega)^r}\int_0^{\infty}exp(-ut)t^{\lambda-1}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)(z_it^{\mu})^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}t$$

Changing the order of integration, which permissible under the stated conditions and applied the formula of Laplace transform , we have :

$$Ł=\frac{u^{-\lambda}}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)\,z_i^{s_i}\,u^{-\mu(s_1+\cdots+s_r)}\,\Gamma(\lambda+\mu(s_1+\cdots+s_r))$$
$\mathrm{d}s_1\cdots\mathrm{d}s_r$

After simple adjustement we finally arrived at (3.1).

Lemme 3,2. From the Lemma3,1, it is clear that

$$L^{-1}\{u^{-\lambda}\aleph(z_1u^{\mu},\cdots,z_ru^{\mu})\}=t^{\lambda-1}\aleph^{0,\mathfrak{n}:V}_{U_{01}:W}\left(\begin{array}{c|c}\frac{z_1}{t} & \\ \cdot & \mathrm{A:C}\\ \cdot & \\ \cdot & (\lambda;\mu,\cdots,\mu),B:D\\ \frac{z_r}{t} & \end{array}\right) \tag{3.2}$$

where $\lambda,z_1,\cdots,z_r,u\in\mathbb{C}$, $Re(u)>0,\mu>0$, $U_{01}=p_i,q_i+1,\tau_i;R$

Theorem 3.3. If $\upsilon>0$, c > 0, d > 0 , $\mu>0$, $Re(s)>|\,d|^{\upsilon/\alpha}$, c # d , then for the solution of the generalized fractional kinetic equation

$$N(t)-N_0t^{\mu-1}\aleph(d^{\upsilon}t^{\upsilon}z_0,\cdots,d^{\upsilon}t^{\upsilon}z_r)=-c^{\upsilon}{}_0D_t^{-\upsilon}N(t) \tag{3.3}$$

there holds the formula :

$$N(t)=N_0t^{\mu-1}\sum_{k=0}^{\infty}(-1)^k(ct)^{k\upsilon}\,\aleph^{0,\mathfrak{n}+1:V}_{U_{11}:W}\left(\begin{array}{c|c}\mathrm{z}_1(dt)^{\upsilon} & \\ \cdot & (1-\mu;\upsilon\cdots\upsilon),A:C\\ \cdot & \\ \cdot & (1-\mathrm{k}\upsilon-\mu;\upsilon\cdots\upsilon),B:D\\ \mathrm{z}_r(dt)^{\upsilon} & \end{array}\right) \tag{3.4}$$

Where $U_{11}=p_i+1,q_i+1,\tau_i;R$

Proof. Taking Laplace transform on the both side of (3.3) and using Lemme 3,1 , we get

$$\overline{N}(\mathrm{s})-\frac{N_0}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)\,\Gamma(\mu+\upsilon\sum_{i=1}^{r}s_i)$$

$$\times \; s^{-\mu-\upsilon\sum_{i=1}^{r}s_i} \; \mathrm{d}s_1\cdots\mathrm{d}s_r = -c^{\upsilon}u^{\upsilon}\;\overline{N}\;(\;\mathrm{s}\;) \tag{3.5}$$

Solving for $\overline{N}\;(\;\mathrm{s}\;)$, it gives

$$\overline{N}\;(\;\mathrm{s}\;) = \frac{N_0}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)\;\Gamma(\mu+\upsilon\sum_{i=1}^{r}s_i)$$

$$\times\; s^{-\mu-\upsilon\sum_{i=1}^{r}s_i}\;\mathrm{d}s_1\cdots\mathrm{d}s_r \times (1+c^{\upsilon}u^{-\upsilon})^{-1} \tag{3.6}$$

Now , taking inverse Laplace transform on both side of (3.6) and using Lemme 3.2 , after little arrangement we finally arrived at the desired result (3.4).

4. Alternative Method for Solving Generalized Fractional Kinetic Equation

In this section , we solve the generalized fractional kinetic Eq. (3.3) by an alternate method by using Sumudu transform.

Lemme4.1 The Sumudu transform of the multivariable $\aleph$-function as follows

$$S\{t^{\lambda-1}\aleph\left(z_1t^{\mu},\cdots,z_rt^{\mu}\right)\} = u^{\lambda-1}\aleph_{U_{10}:W}^{0,\mathrm{n}+1:V}\left(\begin{array}{c|c} \mathrm{z}_1u^{\mu} & (1 - \lambda;\mu,\cdots,\mu),A:C \\ \cdot & \\ \cdot & \\ \cdot & \mathrm{B:D} \\ \mathrm{z}_ru^{\mu} & \end{array}\right) \tag{4.1}$$

where $\lambda, z_1,\cdots,z_r, u \in \mathbb{C}$, $Re(u)>0, \mu>0$, $U_{10} = p_i+1, q_i, \tau_i; R$

Proof. In order to prove (4.1) we first express the multivariable $\aleph$-function occuring on the left hand side of (4.1) in terms of Mellin-Barnes contour integral and apply Sumudu transform integral (say I).

$$I = \int_0^{\infty}(ut)^{\lambda-1}\,I(z_1(ut)^{\mu},\cdots,z_r(ut)^{\mu})e^{-t}\,\mathrm{d}t =$$

$$= \frac{1}{(2\pi\omega)^r}\int_0^{\infty}e^{-t}(ut)^{\lambda-1}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)(z_i(tu)^{\mu})^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}t$$

Use the Gamma function $\int_0^{\infty}t^{x-1}e^{-t}\,\mathrm{d}t = \Gamma(x)$, we arrive at

$$I = \frac{u^{\lambda-1}}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)\;z_i^{s_i}\Gamma(\lambda+\mu s_1+\cdots+\mu s_r)u^{\mu s_1+\cdots+\mu s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r$$

After simple adjustement we finally arrived at (4.1).

Lemme4.2 From the Lemma 4,1, it is clear that

$$S^{-1}\{u^{\lambda}\aleph\left(z_1u^{\mu},\cdots,z_ru^{\mu}\right)\}=t^{\lambda-1}=t^{\lambda-1}\aleph^{0,\mathfrak{n}:V}_{U_{01}:W}\left(\begin{array}{c|c} \mathrm{z}_1t^{\mu} & \mathrm{A}:\mathrm{C} \\ \cdot & \\ \cdot & \\ \cdot & (\lambda;\mu,\cdots,\mu),B:D \\ \mathrm{z}_rt^{\mu} & \end{array}\right) \tag{4.2}$$

where $\lambda,z_1,\cdots,z_r,u\in\mathbb{C}\,,Re(u)>0,\mu>0\,,U_{01}=p_i,q_i+1,\tau_i;R$

Theorem2. If $\upsilon>0$, c > 0, d > 0 , $\mu>0$, $Re(s)>|\,d|^{\upsilon/\alpha}$, c # d , then for the solution of the generalized fractional kinetic equation

$$N(t)-N_0t^{\mu-1}\aleph(d^{\upsilon}t^{\upsilon}z_0,\cdots,d^{\upsilon}t^{\upsilon}z_r)=-c^{\upsilon}{}_0D_t^{-\upsilon}N(t) \tag{4.3}$$

there holds the formula :

$$N(t)=N_0t^{\mu-1}\sum_{k=0}^{\infty}(-1)^k(ct)^{k\upsilon}\,\aleph^{0,\mathfrak{n}+1:V}_{U_{11}:W}\left(\begin{array}{c|c} \mathrm{z}_1(dt)^{\upsilon} & (1-\mu;\upsilon\cdots\upsilon),A:C \\ \cdot & \\ \cdot & \\ \cdot & (1-\mathrm{k}\upsilon-\mu;\upsilon\cdots\upsilon),B:D \\ \mathrm{z}_r(dt)^{\upsilon} & \end{array}\right) \tag{4.4}$$

Where : $U_{11}=p_i+1,q_i+1,\tau_i;R$

Proof. Appling the Sumudu transform on the both side of (4.3) and using Lemme 4.1 , we get

$$\overline{N}\,(\,\mathrm{u}\,)-\frac{N_0}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}(\phi_i(s_i)(d^{\upsilon}z_i)^{s_i})\;\Gamma(\mu+\upsilon\sum_{i=1}^{r}s_i)$$
$$\times\,u^{\mu+\upsilon\sum_{i=1}^{r}s_i-1}\;\mathrm{d}s_1\cdots\mathrm{d}s_r\;=-c^{\upsilon}u^{-\upsilon}\,\overline{N}\,(\,\mathrm{u}\,) \tag{4.5}$$

Solving for $\overline{N}\,(\,\mathrm{u}\,)$, it gives

$$\overline{N}\,(\,\mathrm{u}\,)=\frac{N_0}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}(\phi_i(s_i)(d^{\upsilon}z_i)^{s_i})\;\Gamma(\mu+\upsilon\sum_{i=1}^{r}s_i)$$
$$\times\,u^{\mu+\upsilon\sum_{i=1}^{r}s_i-1}\,\mathrm{d}s_1\cdots\mathrm{d}s_r\times(1+c^{\upsilon}u^{-\upsilon})^{-1} \tag{4.6}$$

Now , taking inverse Sumudu transform both sides of the Eq. (4.6) and using Lemme 4.1 , we get after little arrangement the desired result (4.4).

5. Particular cases

Remarks : If $\tau_i = \tau_{i^{(k)}} = 1$, then the Aleph-function of several variables degenere in the I-function of several variables defined by Sharma and Ahmad [12].

Corollary 5.1 If $\upsilon > 0$, c > 0, d > 0 , $\mu > 0$, $Re(s) > |\, d|^{\upsilon/\alpha}$, c # d , then for the solution of the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1} I(t^{\upsilon} d^{\upsilon} z_1, \cdots, t^{\upsilon} d^{\upsilon} z_r) = -c^{\upsilon} {}_0D_t^{-\upsilon} N(t) \tag{5.1}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{k\upsilon} I_{U_{11}:W}^{0,\mathfrak{n}+1:V} \left(\begin{array}{c|c} z_1(dt)^{\upsilon} & \\ \cdot & (1 - \mu; \upsilon \cdots \upsilon), A : C \\ \cdot & \\ \cdot & (1 - k\upsilon - \mu; \upsilon \cdots \upsilon), B : D \\ z_r(dt)^{\upsilon} & \end{array} \right) \tag{5.2}$$

Where : $U_{11} = p_i + 1, q_i + 1; R$

If $R = R^{(1)} =, \cdots, R^{(r)} = 1$, the multivariable I-function degenere in the multivariable H-function and we have , see Choudhary et al [5].

Corollary 5.2 If $\upsilon > 0$, c > 0, d > 0 , $\mu > 0$, $Re(s) > |\, d|^{\upsilon/\alpha}$, c # d , then for the solution of the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1} H(t^{\upsilon} d^{\upsilon} z_1, \cdots, t^{\upsilon} d^{\upsilon} z_r) = -c^{\upsilon} {}_0D_t^{-\upsilon} N(t) \tag{5.3}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{k\upsilon} H_{p+1,q+1:p_1,q_1;\cdots;p_r,q_r}^{0,n+1:m_1,n_1;\cdots;m_r,n_r} \left(\begin{array}{c|c} z_1(dt)^{\upsilon} & (1 - \mu; \upsilon \cdots \upsilon), (a_{rj}; \alpha'_{rj}, \cdots, \alpha_{rj}^{(r)})_{1,p_r} : (a'_j, \alpha'_j)_{1,p'}; \cdots; (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}} \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r(dt)^{\upsilon} & (1 - k\upsilon - \mu; \upsilon \cdots \upsilon), (b_{rj}; \beta'_{rj}, \cdots, \beta_{rj}^{(r)})_{1,q_r} : (b'_j, \beta'_j)_{1,q'}; \cdots; (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}} \end{array} \right) \tag{5.4}$$

If r = 2 , the multivariable H-function reduces into H-function of two variables ,for more details , see Srivastava et al [13].

Corollary :5.3 If $\upsilon > 0$, c > 0, d > 0 , $\mu > 0$, $Re(s) > |\, d|^{\upsilon/\alpha}$, c # d , then for the solution of the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1} H(d^{\upsilon} t^{\upsilon} z_1, d^{\upsilon} t^{\upsilon} z_2) = -c^{\upsilon} {}_0D_t^{-\upsilon} N(t) \tag{5.5}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{k\upsilon} H^{0,n+1:m_1,n_1;m_2,n_2}_{p+1,q+1:p_1,q_1;p_2,q_2}\left(\begin{matrix} z_1(dt)^{\upsilon} \\ z_2(dt)^{\upsilon} \end{matrix} \middle| \begin{matrix} (1\text{-}\mu;\upsilon,\upsilon),(a_j;\alpha'_j,\alpha''_j)_{1,p} \\ (1\text{-}k\upsilon-\mu;\upsilon,\upsilon),(b_j;\beta'_j,\beta''_j)_{1,q} \end{matrix} \right.$$

$$\left. \begin{matrix} (c'_j;\gamma'_j)_{1,p_1};(c''j;\gamma''j)_{1,p_2} \\ (d'_j;\delta'_j)_{1,q_1};(d''_j;\delta''_j)_{1,q_2} \end{matrix} \right) \tag{5.6}$$

If r = 1 , the multivariable Aleph-function reduces in Aleph-function of one variable and consequently there holds the following result.

Corollary 5.4 If $\upsilon > 0, c > 0, \mu > 0$, then the solution of the equation

$$N(t)-N_0 t^{(\mu-1)} \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(d^{\upsilon} t^{\upsilon} z \;\middle|\; \begin{matrix} (a_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (b_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right) = -c^{\upsilon} {}_0D_t^{-\upsilon} N(t) \tag{5.7}$$

is given :

$$N(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{k\upsilon} \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+1,c_i;r}\left(d^{\upsilon} t^{\upsilon} z \;\middle|\; \begin{matrix} (1\text{-}\mu;\upsilon,\upsilon), \\ (1\text{-}k\upsilon-\mu;\upsilon,\upsilon), \end{matrix} \right.$$

$$\left. \begin{matrix} ,(a_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ ,(b_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right) \tag{5.8}$$

For more details ,see ,Kumar et al [7]

6. Conclusion and remarks

Multivariable Aleph-function is general in nature and includes a number of known and new results as particular cases. This extended fractional kinetic equation can be used to compute the particle reaction rate and may be utilized in other branch of mathematics. Results obtained in this paper provide an extension of [3, 6, 7 , 8].

REFERENCES

[1] F. B. M. Belgacem and A. A. Karaballi, Sumudu transform fundamental properties investigations and applications, International J. Appl. Math.Stoch. Anal., (2005), 1-23.

[2] F. B. M. Belgacem, A. A. Karaballi and S. L Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, Mathematical problems in Engineering, 3(2003), 103-118.

[3] V. B. L. Chaurasia and J. Singh, Application of Sumudu Transform in Fractional Kinetic Equations, Gen. Math. Notes, 2(1) (2011), 86-95.

[4] V. B. L. Chaurasia and J. Singh, Application of Sumudu Transform in Schrodinger Equation Occurring in Quantum

Mechanics, Applied Mathematical Sciences, 4(57)(2010), 2843-2850.

[5] A.Choudhary , D.Kumar and J.Singh , New generalization of fractional kinetic equation using the multivariable H-function , Journal of Fractional Calculus and Applications , Vol 5 (3S) No,.21 ,page 1-9

[6] H. J.Haubold and A. M. Mathai, The fractional kinetic equation and thermonuclear functions, Astrophysics and Space-Sciences, 327 (2000), 53-63.

[7]D .Kumar and J.Choi, Solutions of generalized fractional kinetic equation involving Aleph-function, Mathematical Communications, Vol20(2015) , 113-123.

[8] A. M. Mathai, R. K. Saxena and H. J. Haubold, On fractional kinetic equations, Astrophys. Space Sci., 282(2002), 281-287.

[9] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, (1993).

[10] K. B. Oldhman and J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974.

[11] G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, New York, (1993).

[12] C.K. Sharma and S.S. Ahmad : On the multivariable I-function. Acta ciencia Indica Math , 1992 vol 19

[13] H.M. Srivastava, K.C. Gupta and S.P. Goyal, The H -Functions of One and Two Variables, South Asian Publishers Pvt. Ltd., New Delhi, (1982).

[14] H. M. Srivastava and R. K. Saxena, Operators of fractional integration and their applications, Applied Mathematics and Computation 118, (2001), 1-52.

[15] G. K. Watugala, Sumudu transform- a new integral transform to solve differential equations and control engineering problems, Math. Engg. Indust. 6(4) (1998), 319-329

Personal adress : 411 Avenue Joseph Raynaud

Le parc Fleuri , Bat B

83140 , Six-Fours les plages

Tel : 06-83-12-49-68

Department : VAR

Counry : FRANCE