

# On multiple eulerian integral involving the multivariable Aleph-function

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**ABSTRACT**

Recently, Raina and Srivastava [2] and Srivastava and Hussain [5] have provided closed-form expressions for a number of a general eulerian integrals involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple eulerian integrals involving a multivariable Aleph-function with general arguments. These integrals will serve as a key formula from which one can deduce numerous useful integrals.

Keywords :Multivariable Aleph-function, multiple eulerian integral , Multivariable I-function, Aleph-function of two variables.

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## 1.Introduction and preliminaries.

The object of this document is to evaluate a multiple Eulerian integrals involving the Aleph-function of several variables. These function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [3], itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

$$\begin{aligned} \text{We have : } \aleph(z_1, \dots, z_r) &= \aleph_{p_i, q_i, \tau_i; R: p_i(1), q_i(1), \tau_i(1); R^{(1)}; \dots; p_i(r), q_i(r), \tau_i(r); R^{(r)}}^{0, n: m_1, n_1, \dots, m_r, n_r} \left( \begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \right) \\ & [ (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)})_{1, n} ] , [ \tau_i(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(r)})_{n+1, p_i} ] : \\ & \dots \dots \dots [ \tau_i(b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(r)})_{m+1, q_i} ] : \\ & \left( \begin{matrix} [(c_j^{(1)}), \gamma_j^{(1)})_{1, n_1}], [\tau_{i(1)}(c_{ji(1)}, \gamma_{ji(1)})_{n_1+1, p_i(1)}]; \dots ; [(c_j^{(r)}), \gamma_j^{(r)})_{1, n_r}], [\tau_{i(r)}(c_{ji(r)}, \gamma_{ji(r)})_{n_r+1, p_i(r)}] \\ [(d_j^{(1)}), \delta_j^{(1)})_{1, m_1}], [\tau_{i(1)}(d_{ji(1)}, \delta_{ji(1)})_{m_1+1, q_i(1)}]; \dots ; [(d_j^{(r)}), \delta_j^{(r)})_{1, m_r}], [\tau_{i(r)}(d_{ji(r)}, \delta_{ji(r)})_{m_r+1, q_i(r)}] \end{matrix} \right) \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \zeta_k(s_k) z_k^{s_k} ds_1 \dots ds_r \end{aligned} \tag{1.1}$$

with  $\omega = \sqrt{-1}$

$$\psi(s_1, \dots, s_r) = \frac{\prod_{j=1}^n \Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} s_k)}{\sum_{i=1}^R [\tau_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^r \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^r \beta_{ji}^{(k)} s_k)]} \tag{1.2}$$

$$\text{and } \zeta_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_i^{(k)}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n_k+1}^{p_i^{(k)}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]} \tag{1.3}$$

where  $j = 1$  to  $r$  and  $k = 1$  to  $r$   
 Suppose , as usual , that the parameters  
 $a_j, j = 1, \dots, p; b_j, j = 1, \dots, q;$

$$c_j^{(k)}, j = 1, \dots, n_k; c_{j i^{(k)}}^{(k)}, j = n_k + 1, \dots, p_{i^{(k)}};$$

$$d_j^{(k)}, j = 1, \dots, m_k; d_{j i^{(k)}}^{(k)}, j = m_k + 1, \dots, q_{i^{(k)}};$$

with  $k = 1 \dots, r, i = 1, \dots, R, i^{(k)} = 1, \dots, R^{(k)}$

are complex numbers, and the  $\alpha' s, \beta' s, \gamma' s$  and  $\delta' s$  are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^n \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_{j i}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{j i^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{j i}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{j i^{(k)}}^{(k)} \leq 0 \tag{1.4}$$

The real numbers  $\tau_i$  are positives for  $i = 1$  to  $R$ ,  $\tau_{i^{(k)}}$  are positives for  $i^{(k)} = 1$  to  $R^{(k)}$

The contour  $L_k$  is in the  $s_k$ -p lane and run from  $\sigma - i\infty$  to  $\sigma + i\infty$  where  $\sigma$  is a real number with loop, if necessary, ensure that the poles of  $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$  with  $j = 1$  to  $m_k$  are separated from those of  $\Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(k)} s_k)$  with  $j = 1$  to  $n$  and  $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$  with  $j = 1$  to  $n_k$  to the left of the contour  $L_k$ . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^n \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{j i}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{j i}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{j i^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{j i^{(k)}}^{(k)} > 0, \text{ with } k = 1 \dots, r, i = 1, \dots, R, i^{(k)} = 1, \dots, R^{(k)} \tag{1.5}$$

The complex numbers  $z_i$  are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\alpha_1} \dots |z_r|^{\alpha_r}), \max(|z_1| \dots |z_r|) \rightarrow 0$$

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\beta_1} \dots |z_r|^{\beta_r}), \min(|z_1| \dots |z_r|) \rightarrow \infty$$

where, with  $k = 1, \dots, r : \alpha_k = \min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \dots, m_k$  and

$$\beta_k = \max[Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \dots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R; V = m_1, n_1; \dots; m_r, n_r \tag{1.6}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \dots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)})_{1,n}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}, \tau_{i(1)}(c_{ji(1)}; \gamma_{ji(1)}^{(1)})_{n_1+1,p_{i(1)}}, \dots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}, \tau_{i(r)}(c_{ji(r)}; \gamma_{ji(r)}^{(r)})_{n_r+1,p_{i(r)}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}, \tau_{i(1)}(d_{ji(1)}; \delta_{ji(1)}^{(1)})_{m_1+1,q_{i(1)}}, \dots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}, \tau_{i(r)}(d_{ji(r)}; \delta_{ji(r)}^{(r)})_{m_r+1,q_{i(r)}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \dots, z_r) = \aleph_{p_i, q_i, \tau_i; R: W}^{0, n: V} \left( \begin{array}{c|c} z_1 & \mathbf{A : C} \\ \vdots & \cdot \cdot \cdot \\ \vdots & \mathbf{B : D} \\ z_r & \end{array} \right) \tag{1.12}$$

## 2. Main integral

In this document, we shall establish the following Eulerian multiple integral of multivariable Aleph-function and we shall use the following notations (2.1) and (2.2).

$$\text{Let } f(t_j) = (b_j - a_j) + \rho_j(t_j - a_j) + \sigma_j(b_j - t_j) \tag{2.1}$$

$$g^{(i)}(t_j) = \frac{(t_j - a_j)^{\gamma_j^{(i)}} (b_j - t_j)^{\delta_j^{(i)}} \{f(t_j)\}^{1-\gamma_j^{(i)}-\delta_j^{(i)}}}{\beta_j(b_j - a_j) + (\beta_j \rho_j + \alpha_j - \beta_j)(t_j - a_j) + \beta_j \sigma_j(b_j - t_j)} \tag{2.2}$$

$$j = 1, \dots, n$$

**Formula 1** ([1] p.287)

$$\int_a^b \frac{(t - a)^{\alpha-1} (b - t)^{\beta-1}}{\{b - a + \lambda(t - a) + \mu(b - t)\}^{\alpha+\beta}} dt = \frac{(1 + \lambda)^{-\alpha} (1 + \mu)^{-\beta} \Gamma(\alpha) \Gamma(\beta)}{(b - a) \Gamma(\alpha + \beta)} \tag{2.3}$$

with  $t \in [a; b]$   $a \neq b, Re(\alpha) > 0, Re(\beta) > 0, \eta + \lambda(t - a) + \mu(b - t) \neq 0$

Formula 2

$$\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} \aleph \left( \begin{array}{c|c} z_1 \prod_{j=1}^n [g'(t_j)]^{v'_j} & \\ \vdots & \\ z_r \prod_{j=1}^n [g^{(r)}(t_j)]^{v_j^{(r)}} & \end{array} \right) dt_1 \dots dt_n$$

$$= \prod_{j=1}^n \{(b_j - a_j)^{-1} (1 + \rho_j)^{-\lambda_j - 1} (1 + \sigma_j)^{-\mu_j - 1} \sum_{r_j=0}^{\infty} \frac{\{(\beta_j - \alpha_j)/\beta_j\}^{r_j} (1 + \rho_j)^{-r_j}}{r_j!}$$

$$\aleph_{p_i+3n, q_i+2n, \tau_i; R: W}^{0, n+3n: V} \left( \begin{array}{c|c} z_1 \prod_{j=1}^n \{\beta_j (1 + \rho_j)^{\gamma'_j} (1 + \sigma_j)^{\delta'_j}\}^{-v'_j} & (1-r_1; v'_1, \dots, v_1^{(r)}), \dots, \\ \vdots & \vdots \\ \vdots & \vdots \\ z_r \prod_{j=1}^n \{\beta_j (1 + \rho_j)^{\gamma_j^{(n)}} (1 + \sigma_j)^{\delta_j^{(n)}}\}^{-v_j^{(r)}} & (1; v_1, \dots, v_1^{(r)}), \dots, \end{array} \right)$$

$$\begin{aligned}
 & (1-r_n; v'_n, \dots, v_n^{(r)}), (-\lambda_1 - r_1; \gamma'_1 v'_1, \dots, \gamma_1^{(r)} v_1^{(r)}), (-\mu_1; \delta'_1 v'_1, \dots, \delta_1^{(r)} v_1^{(r)}), \dots, \\
 & \qquad \qquad \qquad \vdots \\
 & \qquad \qquad \qquad \vdots \\
 & (1; v'_n, \dots, v_n^{(r)}), (-\lambda_1 - \mu_1 - r_1 - 1; (\gamma'_1 + \delta'_1) v'_1, \dots, (\gamma_1^{(r)} + \delta_1^{(r)}) v_1^{(r)}), \dots, \\
 & \qquad \qquad \qquad \vdots \\
 & \qquad \qquad \qquad \vdots \\
 & (-\lambda_n - r_n; \gamma'_n v'_n, \dots, \gamma_n^{(r)} v_n^{(r)}), (-\mu_n; \delta'_n v'_n, \dots, \delta_n^{(r)} v_n^{(r)}), A : C \\
 & \qquad \qquad \qquad \vdots \\
 & \qquad \qquad \qquad \vdots \\
 & (-\lambda_n - \mu_n - r_n - 1; (\gamma'_n + \delta'_n) v'_n, \dots, (\gamma_n^{(r)} + \delta_n^{(r)}) v_n^{(r)}), B : D
 \end{aligned} \tag{2.4}$$

Provided that

- a)  $v_j^{(i)} > 0, \gamma_j^{(i)} > 0, \delta_j^{(i)} > 0, \beta_j \neq 0, b_j - a_j \neq 0, \rho_j \neq -1, \sigma_j \neq -1, j = 1, \dots, n, i = 1, \dots, r$
- b)  $(b_j - a_j) + \rho_j(t_j - a_j) + \sigma_j(b_j - t_j) \neq 0, t_j \in [a_j; b_j]$
- c)  $|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi$ , where  $A_i^{(k)}$  is given in (1.5)
- d)  $|(\beta_j - \alpha_j)(t_j - a_j)| < |\beta_j(b_j - a_j) + \rho_j(t_j - a_j) + \sigma_j(b_j - t_j)|$
- e)  $Re[\lambda_j + \sum_{i=1}^r \gamma_j^{(i)} v_j^{(i)} \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] + 1 > 0; Re[\mu_j + \sum_{i=1}^r \delta_j^{(i)} v_j^{(i)} \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] + 1 > 0$   
with  $j = 1, \dots, n, i = 1, \dots, r$
- f) the multiple serie on the R.H.S of (2.4) converges absolutely

**Proof**

Let  $M = \frac{1}{(2\pi\omega)^n} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_n) \prod_{k=1}^r \zeta_k(s_k)$ , we have

$$\begin{aligned}
 & \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} \aleph \begin{pmatrix} z_1 \prod_{j=1}^n [g'(t_j)]^{v'_j} \\ \dots \\ z_r \prod_{j=1}^n [g^{(r)}(t_j)]^{v_j^{(r)}} \end{pmatrix} dt_1 \dots dt_n \\
 & = \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} M \{ \prod_{i=1}^r [z_i^{s_i} \prod_{j=1}^n [g^{(i)}(t_j)]^{v_j^{(i)} s_i}] ds_1 \dots ds_r \} dt_1 \dots dt_n
 \end{aligned}$$

Now, changing the order of multiple integral (wich is justified under the conditions of (2.4)), we find that

$$M \{ \prod_{i=1}^r [z_i^{s_i}] \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} \prod_{j=1}^n [g^{(i)}(t_j)]^{v_j^{(i)} s_i} dt_1 \dots dt_n \} ds_1 \dots ds_r$$

$$\begin{aligned}
 &= M \left\{ \prod_{i=1}^r z_i^{s_i} \prod_{j=1}^n \left[ \int_{a_j}^{b_j} (t_j - a_j)^{\lambda_j + \sum_{i=1}^r \gamma_j^{(i)} v_j^{(i)} s_i} \frac{(b_j - t_j)^{\mu_j + \sum_{i=1}^r \delta_j^{(i)} v_j^{(i)} s_i}}{[f(t_j)]^{\lambda_j + \mu_j + \sum_{i=1}^r (\gamma_j^{(i)} + \delta_j^{(i)}) v_j^{(i)} s_i + 2}} \right. \right. \\
 &\left. \left. \left\{ 1 - \frac{(\beta_j - \alpha_j)(t_j - a_j)}{\beta_j f(t_j)} \right\}^{-\sum_{i=1}^r v_j^{(i)} s_i} dt_j \right] \right\} ds_1 \cdots ds_r \tag{2.5}
 \end{aligned}$$

If  $|(\beta_j - \alpha_j)(t_j - a_j)| < |\beta_j f(t_j)|$ , then we can use binomial expansion and we thus find from (2.5)

$$\begin{aligned}
 &\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} \mathfrak{K} \left( \begin{matrix} z_1 \prod_{j=1}^n [g'(t_j)]^{v_j'} \\ \dots \\ z_r \prod_{j=1}^n [g^{(r)}(t_j)]^{v_j^{(r)}} \end{matrix} \right) dt_1 \cdots dt_n \\
 &= \prod_{j=1}^n \sum_{r_j=0}^{\infty} \frac{\{(\beta_j - \alpha_j)/\beta_j\}^{r_j}}{r_j!} M \left\{ \prod_{i=1}^r [z_i^{s_i} \beta_j^{-\sum_{i=1}^r v_j^{(i)} \Gamma(r_j + \sum_{i=1}^n v_j^{(i)} s_i)} \frac{\Gamma(r_j + \sum_{i=1}^n v_j^{(i)} s_i)}{\Gamma(\sum_{i=1}^n v_j^{(i)} s_i)} \right. \\
 &\left. \int_{a_j}^{b_j} \frac{(t_j - a_j)^{\lambda_j + r_j + \sum_{i=1}^r \gamma_j^{(i)} v_j^{(i)} s_i}}{[f(t_j)]^{\lambda_j + \mu_j + \sum_{i=1}^r (\gamma_j^{(i)} + \delta_j^{(i)}) v_j^{(i)} s_i + 2}} (b_j - t_j)^{\mu_j + \sum_{i=1}^r \delta_j^{(i)} v_j^{(i)} s_i} dt_j \right\} ds_1 \cdots ds_r \tag{2.6}
 \end{aligned}$$

provided that the order of summation and integration can be inverted. Now evaluating the inner-integral in (2.6) with the help of equation (2.1). We finally obtain the formula (2.4)

### 3. Particular cases

**a)** For  $n = 1$ , the equation (2.4) reduces in the following formula after making slight ajustement in parameters.

$$\begin{aligned}
 &\int_a^b \frac{(t - a)^\lambda (b - t)^\mu}{[f(t)]^{\lambda + \mu + 2}} \mathfrak{K} \left( \begin{matrix} z_1 [g'(t)]^{v'} \\ \dots \\ z_r [g^{(r)}(t)]^{v^{(r)}} \end{matrix} \right) dt_1 \cdots dt_n \\
 &= \{(b - a)^{-1} (1 + \rho)^{-\lambda - 1} (1 + \sigma)^{-\mu - 1} \sum_{r'=0}^{\infty} \frac{\{(\beta - \alpha)/\beta\}^{r'} (1 + \rho)^{-r'}\}}{r'!} \\
 &\mathfrak{K}_{p_i+3, q_i+2, \tau_i; R:W}^{0, n+3:V} \left( \begin{matrix} z_1 \{\beta(1 + \rho)^\gamma (1 + \sigma)^\delta\}^{-v'} \\ \vdots \\ z_r \{\beta(1 + \rho)^\gamma (1 + \sigma)^\delta\}^{-v^{(r)}} \end{matrix} \middle| \begin{matrix} (1 - r'; v'_1, \dots, v_1^{(r)}), \\ \dots \\ (1; v'_1, \dots, v_1^{(r)}), \end{matrix} \right) \\
 &\left( \begin{matrix} (-\lambda - r'; \gamma' v', \dots, \gamma^{(r)} v^{(r)}), (-\mu; \delta' v', \dots, \delta^{(r)} v^{(r)}), A : C \\ \vdots \\ (-\lambda - \mu - r' - 1; (\gamma' + \delta') v', \dots, (\gamma^{(r)} + \delta^{(r)}) v^{(r)}), B : D \end{matrix} \right) \tag{3.1}
 \end{aligned}$$

which holds true under the same conditions from (2.4) with  $n = 1$

**b)** Taking  $\beta_j = \alpha_j, j = 1, \dots, n$  in the formula (2.4), we get

$$\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} \aleph \left( \begin{matrix} z_1 \prod_{j=1}^n [g'(t_j)]^{v'_j} \\ \dots \\ z_r \prod_{j=1}^n [g^{(r)}(t_j)]^{v_j^{(r)}} \end{matrix} \right) dt_1 \dots dt_n$$

$$= \prod_{j=1}^n \{(b_j - a_j)^{-1} (1 + \rho_j)^{-\lambda_j - 1} (1 + \sigma_j)^{-\mu_j - 1}\}$$

$$\aleph_{p_i+2n, q_i+n, \tau_i; R:W}^{0, n+2n:V} \left( \begin{matrix} z_1 \prod_{j=1}^n \{\beta_j (1 + \rho_j)^{\gamma'_j} (1 + \sigma_j)^{\delta'_j}\}^{-v'_j} \\ \vdots \\ z_r \prod_{j=1}^n \{\beta_j (1 + \rho_j)^{\gamma_j^{(r)}} (1 + \sigma_j)^{\delta_j^{(r)}}\}^{-v_j^{(r)}} \end{matrix} \right)$$

$$(-\lambda_1; \gamma'_1 v'_1, \dots, \gamma_1^{(r)} v_1^{(r)}), (-\mu_1; \delta'_1 v'_1, \dots, \delta_1^{(r)} v_1^{(r)}), \dots,$$

$$\dots$$

$$(-\lambda_1 - \mu_1 - 1; (\gamma'_1 + \delta'_1) v'_1, \dots, (\gamma_1^{(r)} + \delta_1^{(r)}) v_1^{(r)}), \dots,$$

$$(-\lambda_n -; \gamma'_n v'_n, \dots, \gamma_n^{(r)} v_n^{(r)}), (-\mu_n; \delta'_n v'_n, \dots, \delta_n^{(r)} v_n^{(r)}), A : C$$

$$\dots$$

$$(-\lambda_n - \mu_n - 1; (\gamma'_n + \delta'_n) v'_n, \dots, (\gamma_n^{(r)} + \delta_n^{(r)}) v_n^{(r)}), B : D \quad (3.2)$$

which holds true under the same conditions from (2.4)

**c)** For  $\sigma = \rho = 0$  and  $z_i = (b - t)^{\gamma + \delta - 1} v^{(i)}$ , (3.1) becomes

$$\int_a^b \frac{(t - a)^\lambda (b - t)^\mu}{[(b - a)]^{\lambda + \mu + 2}} \aleph \left( \begin{matrix} z_1 \{(b - a)/\beta\}^{v'} \\ \dots \\ z_r \{(b - a)/\beta\}^{v^{(r)}} \end{matrix} \right) dt_1 \dots dt_n$$

$$= \{(b - a)^{-1} \sum_{r'=0}^{\infty} \frac{\{(\beta - \alpha)/\beta\}^{r'}}{r'!}\} \aleph_{p_i+3, q_i+2, \tau_i; R:W}^{0, n+3:V} \left( \begin{matrix} \{(b-a)/\beta\}^{(\gamma+\delta-1)v'} \\ \vdots \\ \{(b-a)/\beta\}^{(\gamma+\delta-1)v^{(r)}} \end{matrix} \right)$$

$$\left. \begin{aligned} & (1-r'; v'_1, \dots, v_1^{(r)}), (-\lambda - r'; \gamma'v', \dots, \gamma^{(r)}v^{(r)}), (-\mu; \delta'v', \dots, \delta^{(r)}v^{(r)}), A : C \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & (1 ; v'_1, \dots, v_1^{(r)}), (-\lambda - \mu - r' - 1; (\gamma' + \delta')v', \dots, (\gamma^{(r)} + \delta^{(r)})v^{(r)}), B : D \end{aligned} \right) \quad (3.3)$$

which holds true under the same conditions from (2.4) with  $n = 1$

4; Multivariable I-function

Let  $\tau_i = \tau_{i(1)} = \dots = \tau_{i(r)} = 1$ , the multivariable Aleph-function degenerate to multivariable I-function defined by Sharma et al [3].

$$\begin{aligned} & \text{a) } \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} I \left( \begin{matrix} z_1 \prod_{j=1}^n [g'(t_j)]^{v'_j} \\ \dots \\ z_r \prod_{j=1}^n [g^{(r)}(t_j)]^{v_j^{(r)}} \end{matrix} \right) dt_1 \dots dt_n \\ & = \prod_{j=1}^n \{ (b_j - a_j)^{-1} (1 + \rho_j)^{-\lambda_j - 1} (1 + \sigma_j)^{-\mu_j - 1} \sum_{r_j=0}^{\infty} \frac{\{(\beta_j - \alpha_j)/\beta_j\}^{r_j} (1 + \rho_j)^{-r_j}}{r_j!} \\ & \quad I_{\substack{0, n+3n:V \\ p_i+3n, q_i+2n; R:W}} \left( \begin{matrix} z_1 \prod_{j=1}^n \{ \beta_j (1 + \rho_j)^{\gamma'_j} (1 + \sigma_j)^{\delta'_j} \}^{-v'_j} \\ \vdots \\ z_r \prod_{j=1}^n \{ \beta_j (1 + \rho_j)^{\gamma_j^{(n)}} (1 + \sigma_j)^{\delta_j^{(n)}} \}^{-v_j^{(r)}} \end{matrix} \middle| \begin{matrix} (1-r_1; v'_1, \dots, v_1^{(r)}), \dots, \\ \vdots \\ (1 ; v'_1, \dots, v_1^{(r)}), \dots, \end{matrix} \right) \end{aligned}$$

$$\left. \begin{aligned} & (1-r_n; v'_n, \dots, v_n^{(r)}), (-\lambda_1 - r_1; \gamma'_1 v'_1, \dots, \gamma_1^{(r)} v_1^{(r)}), (-\mu_1; \delta'_1 v'_1, \dots, \delta_1^{(r)} v_1^{(r)}), \dots, \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & (1 ; v'_n, \dots, v_n^{(r)}), (-\lambda_1 - \mu_1 - r_1 - 1; (\gamma'_1 + \delta'_1)v'_1, \dots, (\gamma_1^{(r)} + \delta_1^{(r)})v_1^{(r)}), \dots, \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & (-\lambda_n - r_n; \gamma'_n v'_n, \dots, \gamma_n^{(r)} v_n^{(r)}), (-\mu_n; \delta'_n v'_n, \dots, \delta_n^{(r)} v_n^{(r)}), A : C \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & \qquad \qquad \qquad \dots \qquad \qquad \qquad \dots \\ & (-\lambda_n - \mu_n - r_n - 1; (\gamma'_n + \delta'_n)v'_n, \dots, (\gamma_n^{(r)} + \delta_n^{(r)})v_n^{(r)}), B : D \end{aligned} \right) \quad (4.1)$$

which holds true under the same conditions from (2.4)

b) Taking  $\beta_j = \alpha_j, j = 1, \dots, n$  in the formula (4.1), we get

$$\begin{aligned} & \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} I \left( \begin{matrix} z_1 \prod_{j=1}^n [g'(t_j)]^{v'_j} \\ \dots \\ z_r \prod_{j=1}^n [g^{(r)}(t_j)]^{v_j^{(r)}} \end{matrix} \right) dt_1 \dots dt_n \\ & = \prod_{j=1}^n \{ (b_j - a_j)^{-1} (1 + \rho_j)^{-\lambda_j - 1} (1 + \sigma_j)^{-\mu_j - 1} \} \end{aligned}$$

$$I_{p_i+2n, q_i+n; R:W}^{0, n+2n:V} \left( \begin{array}{c} z_1 \prod_{j=1}^n \{ \beta_j (1 + \rho_j)^{\gamma'_j} (1 + \sigma_j)^{\delta'_j} \}^{-v'_j} \\ \vdots \\ z_r \prod_{j=1}^n \{ \beta_j (1 + \rho_j)^{\gamma_j^{(n)}} (1 + \sigma_j)^{\delta_j^{(n)}} \}^{-v_j^{(r)}} \end{array} \right)$$

$$(-\lambda_1; \gamma'_1 v'_1, \dots, \gamma_1^{(r)} v_1^{(r)}), (-\mu_1; \delta'_1 v'_1, \dots, \delta_1^{(r)} v_1^{(r)}), \dots,$$

$$\vdots$$

$$(-\lambda_1 - \mu_1 - 1; (\gamma'_1 + \delta'_1) v'_1, \dots, (\gamma_1^{(r)} + \delta_1^{(r)}) v_1^{(r)}), \dots,$$

$$\left( \begin{array}{c} (-\lambda_n - 1; \gamma'_n v'_n, \dots, \gamma_n^{(r)} v_n^{(r)}), (-\mu_n; \delta'_n v'_n, \dots, \delta_n^{(r)} v_n^{(r)}), A : C \\ \vdots \\ (-\lambda_n - \mu_n - 1; (\gamma'_n + \delta'_n) v'_n, \dots, (\gamma_n^{(r)} + \delta_n^{(r)}) v_n^{(r)}), B : D \end{array} \right) \tag{4.2}$$

which holds true under the same conditions from (2.4)

c) For  $n = 1, \beta = \alpha,$  and  $z_i = (b - t)^{\gamma+\delta-1} v^{(i)},$  (4.1) becomes

$$\int_a^b \frac{(t - a)^\lambda (b - t)^\mu}{[(b - a)]^{\lambda+\mu+2}} I \left( \begin{array}{c} z_1 \{ (b - a) / \beta \}^{v'} \\ \dots \\ z_r \{ (b - a) / \beta \}^{v^{(r)}} \end{array} \right) dt_1 \dots dt_n$$

$$= \{ (b - a)^{-1} \sum_{r'=0}^{\infty} \frac{\{ (\beta - \alpha) / \beta \}^{r'}}{r'!} I_{p_i+3, q_i+2; R:W}^{0, n+3:V} \left( \begin{array}{c} \{ (b-a) / \beta \}^{(\gamma+\delta-1)v'} \\ \vdots \\ \{ (b-a) / \beta \}^{(\gamma+\delta-1)v^{(r)}} \end{array} \right)$$

$$\left( \begin{array}{c} (1-r'; v'_1, \dots, v_1^{(r)}), (-\lambda - r'; \gamma' v', \dots, \gamma^{(r)} v^{(r)}), (-\mu; \delta' v', \dots, \delta^{(r)} v^{(r)}), A : C \\ \vdots \\ (1; v'_1, \dots, v_1^{(r)}), (-\lambda - \mu - r' - 1; (\gamma' + \delta') v', \dots, (\gamma^{(r)} + \delta^{(r)}) v^{(r)}), B : D \end{array} \right) \tag{4.3}$$

which holds true under the same conditions from (2.4) with  $n = 1$

### 5. Aleph-function of two variables

In these section,  $r = 2$  and we obtain the Aleph-function of two variables defined by K. Sharma [4].





$$\begin{aligned}
 &(-\lambda_1 - r_1; \gamma'_1 v'_1, \gamma_1^{(2)} v_1^{(2)}), (-\mu_1; \delta'_1 v'_1, \delta_1^{(2)} v_1^{(2)}), \\
 &\quad \vdots \\
 &(-\lambda_1 - \mu_1 - r_1 - 1; (\gamma'_1 + \delta'_1) v'_1, (\gamma_1^{(2)} + \delta_1^{(2)}) v_1^{(2)}), \\
 & \left. \begin{aligned}
 &(-\lambda_2 - r_2; \gamma'_2 v'_2, \gamma_2^{(2)} v_2^{(2)}), (-\mu_2; \delta'_2 v'_2, \delta_2^{(2)} v_2^{(2)}), A : C \\
 &\quad \vdots \\
 &(-\lambda_2 - \mu_2 - r_2 - 1; (\gamma'_2 + \delta'_2) v'_2, (\gamma_2^{(2)} + \delta_2^{(2)}) v_2^{(2)}), B : D
 \end{aligned} \right) \tag{5.2}
 \end{aligned}$$

which holds true under the same conditions from (2.4) with  $r = 2$

c) Taking  $\beta_j = \alpha_j, j = 1, \dots, n$  in the formula (5.1), we get

$$\begin{aligned}
 &\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^n \frac{(t_j - a_j)^{\lambda_j} (b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j + \mu_j + 2}} \aleph \left( \begin{matrix} z_1 \prod_{j=1}^n [g'(t_j)]^{v'_j} \\ \dots \\ z_2 \prod_{j=1}^n [g^{(2)}(t_j)]^{v_j^{(2)}} \end{matrix} \right) dt_1 \dots dt_n \\
 &= \prod_{j=1}^n \{ (b_j - a_j)^{-1} (1 + \rho_j)^{-\lambda_j - 1} (1 + \sigma_j)^{-\mu_j - 1} \} \\
 & \aleph_{p_i+2n, q_i+n, \tau_i; R; W}^{0, n+2n:V} \left( \begin{matrix} z_1 \prod_{j=1}^n \{ \beta_j (1 + \rho_j)^{\gamma'_j} (1 + \sigma_j)^{\delta'_j} \}^{-v'_j} \\ \vdots \\ z_2 \prod_{j=1}^n \{ \beta_j (1 + \rho_j)^{\gamma_j^{(n)}} (1 + \sigma_j)^{\delta_j^{(n)}} \}^{-v_j^{(2)}} \end{matrix} \right)
 \end{aligned}$$

$$\begin{aligned}
 &(-\lambda_1; \gamma'_1 v'_1, \gamma_1^{(2)} v_1^{(2)}), (-\mu_1; \delta'_1 v'_1, \delta_1^{(2)} v_1^{(2)}), \dots, \\
 &\quad \vdots \\
 &(-\lambda_1 - \mu_1 - 1; (\gamma'_1 + \delta'_1) v'_1, (\gamma_1^{(2)} + \delta_1^{(2)}) v_1^{(2)}), \dots, \\
 & \left. \begin{aligned}
 &(-\lambda_n -; \gamma'_n v'_n, \gamma_n^{(2)} v_n^{(2)}), (-\mu_n; \delta'_n v'_n, \delta_n^{(2)} v_n^{(2)}), A : C \\
 &\quad \vdots \\
 &(-\lambda_n - \mu_n - 1; (\gamma'_n + \delta'_n) v'_n, (\gamma_n^{(2)} + \delta_n^{(2)}) v_n^{(2)}), B : D
 \end{aligned} \right) \tag{5.3}
 \end{aligned}$$

which holds true under the same conditions from (2.4) with  $r = 2$

d) For  $n = 1, \beta_j = \alpha_j, j = 1, \dots, n$ , and  $z_i = (b - t)^{\gamma + \delta - 1} v^{(i)}$ , (5.1) becomes

$$\int_a^b \frac{(t - a)^\lambda (b - t)^\mu}{[(b - a)]^{\lambda + \mu + 2}} \aleph \left( \begin{matrix} z_1 \{ (b - a) / \beta \}^{v'} \\ \dots \\ z_2 \{ (b - a) / \beta \}^{v^{(2)}} \end{matrix} \right) dt_1 \dots dt_n$$

$$= \{ (b-a)^{-1} \sum_{r'=0}^{\infty} \frac{\{(\beta-\alpha)/\beta\}^{r'}}{r'!} \}_{p_i+3, q_i+2, \tau_i; R:W} \left( \begin{array}{c} \{(b-a)/\beta\}^{(\gamma+\delta-1)v'} \\ \vdots \\ \{(b-a)/\beta\}^{(\gamma+\delta-1)v^{(2)}} \end{array} \left| \begin{array}{c} (1-r' ; v'_1, v_1^{(2)}), \\ \vdots \\ (1 ; v'_1, v_1^{(2)}), \end{array} \right. \right. \\
 \left. \left. \begin{array}{c} (-\lambda - r' ; \gamma'v', \gamma^{(2)}v^{(2)}), (-\mu ; \delta'v', \delta^{(2)}v^{(2)}), A : C \\ \vdots \\ (-\lambda - \mu - r' - 1 ; (\gamma' + \delta')v', (\gamma^{(2)} + \delta^{(2)})v^{(2)}), B : D \end{array} \right) \right. \tag{5.4}$$

**6. Conclusion**

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as , multivariable H-function , defined by Srivastava et al [6] , the Aleph-function of two variables defined by K.sharma [4].

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