

FINITE DOUBLE SUMS OF KAMPÉ DE FÉRIET'S DOUBLE HYPERGEOMETRIC FUNCTION OF HIGHER ORDER

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Abstract

In this paper we obtain four finite double sums of Kampé de Fériet's double hypergeometric function of higher order using Gauss's summation theorem. Associated and contiguous relations for Kummer's confluent hypergeometric function, Gauss's ordinary hypergeometric function, Saalschütz's summation theorem, double sums involving Appell, Humbert, Karlsson, Bessel functions, are obtained as special cases of our main double summations.

Keywords and Phrases:

Gauss's summation theorem, Saalschütz's summation theorem, Karlsson's quadruple hypergeometric function, Bessel's function, Kampé de Fériet's double hypergeometric function.

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1 Introduction and Preliminaries

Throughout in present paper, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \dots\}, \mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\} \text{ and } \mathbb{Z}^- := \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Here, as usual, \mathbb{Z} denotes the set of integers, \mathbb{R} denotes the set of real numbers, \mathbb{R}^+ denotes the set of positive real numbers and \mathbb{C} denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is defined, in terms of the

familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \dots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases}$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Kampé de Fériet's double hypergeometric function of higher order in the modified notation of Srivastava and Panda [8,pp. 423(26), 424(27)], is given by

$$F_{j:m;n}^{p:q;k} \left[\begin{array}{l} (a_p) : (b_q); (d_k); \\ (g_j) : (e_m); (h_n); \end{array} x, y \right] = \sum_{r,s=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_{r+s} \prod_{i=1}^q (b_i)_r \prod_{i=1}^k (d_i)_s}{\prod_{i=1}^j (g_i)_{r+s} \prod_{i=1}^m (e_i)_r \prod_{i=1}^n (h_i)_s} \frac{x^r y^s}{r! s!} \quad (1.1)$$

where (a_p) abbreviates the array of p parameters given by a_1, a_2, \dots, a_p with similar interpretations for $(b_q), (d_k)$ et cetera and for convergence of double hypergeometric series(1.1), we have

(i) $p + q < j + m + 1, p + k < j + n + 1, |x| < \infty$ and $|y| < \infty$ or

(ii) $p + q = j + m + 1, p + k = j + n + 1$, and

$$\begin{cases} |x|^{\frac{1}{p-j}} + |y|^{\frac{1}{p-j}} < 1 & \text{if } p > j \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq j \end{cases}$$

The Appell's double hypergeometric functions F_1, F_2, F_3 and F_4 [3,p.224(5.7.1.6,...,5.7.1.9)] are denoted by $F_{1:0;0}^{1:1;1}, F_{0:1;1}^{1:1;1}, F_{1:0;0}^{0:2;2}$ and $F_{0:1;1}^{2:0;0}$ respectively.

In our investigation we shall use the following results:

(i) Algebraic properties of Pochhammer's symbol

$$(a)_{p+q} = (a)_p (a+p)_q = (a)_q (a+q)_p \quad (1.2)$$

$$(b)_{-k} = \frac{(-1)^k}{(1-b)_k}, \quad (b \neq 0, \pm 1, \pm 2, \pm 3, \dots \text{ and } k = 1, 2, 3, \dots) \quad (1.3)$$

$$(n-k)! = \frac{(-1)^k (n)!}{(-n)_k}, \quad (0 \leq k \leq n) \quad (1.4)$$

(ii) Gauss's summation theorem [7,p.49(Th. 18)]

$${}_2F_1 \left[\begin{matrix} a, b; & 1 \\ c; & \end{matrix} \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (1.5)$$

where $Re(c-a-b) > 0$ and $c \neq 0, -1, -2, -3, -4, \dots$

(iii) Karlsson's quadruple hypergeometric function $H_c^{(4)}$ [5,p. 37(2.1)]

$$H_c^{(4)} [a_1, a_2, a_3, a_4; c; x_1, x_2, x_3, x_4] = \sum_{k_1, k_2, k_3, k_4=0}^{\infty} \frac{(a_1)_{k_4+k_1} (a_2)_{k_1+k_2} (a_3)_{k_2+k_3} (a_4)_{k_3+k_4}}{(c)_{k_1+k_2+k_3+k_4}} \frac{x_1^{k_1}}{(k_1)!} \frac{x_2^{k_2}}{(k_2)!} \frac{x_3^{k_3}}{(k_3)!} \frac{x_4^{k_4}}{(k_4)!} \quad (1.6)$$

(iv) Binomial coefficient

$$\binom{n}{r} = \frac{(n)!}{(r)!(n-r)!} \quad (1.7)$$

Any values of parameters and variables leading to the results given in sections 2 and 4 which do not make sense, are tacitly excluded.

2 Four Finite Double Summations

In this section we obtain the following four finite double summation formulae with the help of series manipulations.

If the values of a and b are adjusted in such a way that $(1-b)$, $(1+a-b)$ are not integers; c is neither zero nor a negative integer then

$$\begin{aligned} & \sum_{r=0}^n \sum_{s=0}^m C(m, n, r, s, a, b, c) F_Q^{P+1:D;E}_{:G;H+1} \left[\begin{matrix} (p_P), a+r : (d_D); (e_E) & ; & x, y \\ (q_Q) & : (g_G); (h_H), c+n+s; & \end{matrix} \right] \\ &= \frac{(b)_{m+n}}{(c)_{m+n}} F_{Q+1:G;H+2}^{P+2:D;E+1} \left[\begin{matrix} (p_P), a, b+n : (d_D); (e_E), b+m+n & ; & x, y \\ (q_Q), b & : (g_G); (h_H), b+n, c+m+n; & \end{matrix} \right] \quad (2.1) \end{aligned}$$

$$\begin{aligned} & \sum_{r=0}^n \sum_{s=0}^m C(m, n, r, s, a, b, c) F_{Q+1:G;H}^{P+1:D+1;E} \left[\begin{matrix} (p_P) & : (d_D), a+r; (e_E); & x, y \\ (q_Q), c+n+s & : (g_G); (h_H); & \end{matrix} \right] \\ &= \frac{(b)_{m+n}}{(c)_{m+n}} F_{Q+2:G+1;H}^{P+1:D+2;E} \left[\begin{matrix} (p_P), b+m+n & : (d_D), a, b+n; (e_E); & x, y \\ (q_Q), b+n, c+m+n & : (g_G), b & ; (h_H); & \end{matrix} \right] \quad (2.2) \end{aligned}$$

$$\sum_{r=0}^n \sum_{s=0}^m C(m, n, r, s, a, b, c) F_{Q: G; H+1}^{P:D+1; E} \left[\begin{array}{l} (p_P) : (d_D), a + r; (e_E) \\ (q_Q) : (g_G); (h_H), c + n + s; \end{array} ; x, y \right] \\ = \frac{(b)_{m+n}}{(c)_{m+n}} F_{Q:G+1;H+2}^{P:D+2;E+1} \left[\begin{array}{l} (p_P) : (d_D), a, b + n; (e_E), b + m + n \\ (q_Q) : (g_G), b; (h_H), b + n, c + m + n; \end{array} ; x, y \right] \quad (2.3)$$

$$\sum_{r=0}^n \sum_{s=0}^m C(m, n, r, s, a, b, c) F_{Q+1:G;H}^{P+1:D;E} \left[\begin{array}{l} (p_P), a + r : (d_D); (e_E); \\ (q_Q), c + n + s : (g_G); (h_H); \end{array} x, y \right] \\ = \frac{(b)_{m+n}}{(c)_{m+n}} F_{Q+2:G;H}^{P+2:D;E} \left[\begin{array}{l} (p_P), b + m + n, a : (d_D); (e_E); \\ (q_Q), c + m + n, b : (g_G); (h_H); \end{array} x, y \right] \quad (2.4)$$

where

$$C(m, n, r, s, a, b, c) = \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r (c-b)_s}{(a-b+1)_{r-n} (c)_{n+s}} \quad (2.5)$$

and Pochhammer symbols and Gamma functions are well defined; denominator parameters in hypergeometric notations are neither zero nor negative integers.

3 Proofs of Finite Double Summations

On expressing each Kampé de Fériet's double hypergeometric function of left hand sides of (2.1), (2.2), (2.3) and (2.4) in power series forms with the help of (1.1) and (2.5), interchanging the order of summations with the help of (1.2), (1.3) and (1.4) and successive applications of Gauss's summation theorem (1.5), we get the right hand sides of (2.1), (2.2), (2.3) and (2.4).

4 Special Cases

Making suitable adjustment of parameters in (2.1), (2.2), (2.3) and (2.4) we can find a number of finite double sums involving double hypergeometric functions of Appell and Humbert [3,pp. 224-226].

Putting $P = E = 1$, $Q = D = G = H = 0$ in (2.2) and using a result of Carlson [1,p. 222(4)], we have following double sums

$$\begin{aligned} & \sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r (c-b)_s}{(a-b+1)_{r-n} (c)_{n+s}} F_{1:1;0}^{2:1;0} \left[\begin{array}{l} p, a+r+e : a+r ; -; \\ c+n+s : a+r+e ; -; \end{array} x-y, y \right] \\ & = \frac{(b)_{m+n}}{(c)_{m+n}} F_{2:1;0}^{2:2;1} \left[\begin{array}{l} p, b+m+n : a, b+n; e ; \\ b+n, c+m+n : b ; -; \end{array} x, y \right] \end{aligned} \quad (4.1)$$

When $y = x$ and $e = 0$, (4.1) reduces to a known result of Pathan [6,p.58(2.1)] which was obtained with the help of operational calculus technique

$$\sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r (c-b)_s}{(a-b+1)_{r-n} (c)_{n+s}} {}_2F_1 \left[\begin{array}{l} a+r, p ; \\ c+n+s ; \end{array} x \right] = \frac{(b)_{m+n}}{(c)_{m+n}} {}_3F_2 \left[\begin{array}{l} a, b+m+n, p ; \\ b, c+m+n ; \end{array} x \right] \quad (4.2)$$

where ${}_2F_1$ and ${}_3F_2$ are Gauss's ordinary hypergeometric function [7,p.45(1)] and Clausen's hypergeometric function [7,p.73(2)], respectively.

In (4.2) putting $n = 0, m = 1, b = a$ and $c = d - 1$, we get a known contiguous relation [7,p.71,Ex. 21(2,13)]

$$(a-d+1) {}_2F_1 \left[\begin{array}{l} a, p ; \\ d ; \end{array} x \right] = a {}_2F_1 \left[\begin{array}{l} a+1, p ; \\ d ; \end{array} x \right] - (d-1) {}_2F_1 \left[\begin{array}{l} a, p ; \\ d-1 ; \end{array} x \right] \quad (4.3)$$

When $m = 0, x = 1$ and $p = b$ in (4.2) and using Gauss's summation theorem (1.5), we get Saalschütz's summation theorem [7,p.87(Th. 29)] in the following form

$${}_3F_2 \left[\begin{array}{l} -n, a, 1+a-c-n ; \\ 1+a-b-n, 1+a+b-c-n ; \end{array} 1 \right] = \frac{(c-b)_n (b)_n}{(b-a)_n (c-a-b)_n} \quad (4.4)$$

because sum of its denominator parameters exceeds the sum of its numerator parameters by unity.

Putting $P = Q = H = D = 0, G = E = 1$ and setting $e_1 = b + n - 1, g_1 = a, c = 1 + b$ in (2.3), and using a result of Karlsson [4,p.197(8)] for ${}_2F_2$, we get a result involving the product

of two Kummer's confluent hypergeometric functions ${}_1F_1$ [7,p.123(1)]

$$\sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r s!}{(a-b+1)_{r-n} (1+b)_{n+s}} {}_1F_1 \left[\begin{matrix} a+r; \\ a \end{matrix}; x \right] {}_1F_1 \left[\begin{matrix} b+n-1; \\ 1+b+n+s; \end{matrix}; y \right] = \frac{b}{m+1}$$

$${}_1F_1 \left[\begin{matrix} b+n; \\ b \end{matrix}; x \right] {}_1F_1 \left[\begin{matrix} b+n-1; \\ b+n \end{matrix}; y \right] + \frac{b(1-b-n)}{(m+1)(b+m+n)} {}_1F_1 \left[\begin{matrix} b+n; \\ b \end{matrix}; x \right] {}_1F_1 \left[\begin{matrix} b+m+n; \\ 1+b+m+n; \end{matrix}; y \right] \quad (4.5)$$

When $P = Q = E = H = D = G = 0$ in (2.3) and $y = -\frac{z^2}{4}$ we get

$$\sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r (c-b)_s}{(a-b+1)_{r-n} (1-x)^r} \left(\frac{2}{z}\right)^s J_{c+n+s-1}(z)$$

$$= (1-x)^a \left(\frac{z}{2}\right)^{c+n-1} \frac{(b)_{m+n}}{\Gamma(c+m+n)} {}_2F_1 \left[\begin{matrix} a, b+n; \\ b \end{matrix}; x \right] {}_1F_2 \left[\begin{matrix} b+m+n; \\ b+n, c+m+n; \end{matrix}; -\frac{z^2}{4} \right] \quad (4.6)$$

where $J_m(z)$ is Bessel function of first kind of order m [7,p. 108(1)].

When $m = 0$, (4.2) reduces to

$$\sum_{r=0}^n \binom{n}{r} \frac{(-1)^{n+r} (a)_r}{(a-b+1)_{r-n}} {}_2F_1 \left[\begin{matrix} a+r, p; \\ c+n \end{matrix}; x \right] = (b)_n {}_3F_2 \left[\begin{matrix} a, b+n, p; \\ b, c+n \end{matrix}; x \right] \quad (4.7)$$

In (4.5), replacing x and y by xt and yt respectively, multiplying both the sides by $e^{-t} t^{c-1}$ and integrating term by term with respect to t from 0 to ∞ , we have

$$\sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r s!}{(a-b+1)_{r-n} (1+b)_{n+s}} F_2[c; a+r, b+n-1; a, 1+b+n+s; x, y]$$

$$= \frac{b(1-b-n)}{(m+1)(b+m+n)} F_2[c; b+n, b+m+n; b, 1+b+m+n; x, y]$$

$$+ \frac{b}{m+1} F_2[c; b+n, b+n-1; b, b+n; x, y] \quad (4.8)$$

where F_2 is Appell's double hypergeometric function of second kind [3,p. 224(7)].

In (4.8), replacing x and y by xt and $y(1-t)$ respectively, multiplying both the sides by

$t^{k-1}(1-t)^{c-k-1}$ and integrating term by term with respect to t from 0 to 1, we have

$$\begin{aligned} & \sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r s!}{(a-b+1)_{r-n} (1+b)_{n+s}} {}_2F_1 \left[\begin{matrix} k, a+r; & x \\ a & ; \end{matrix} \right] {}_2F_1 \left[\begin{matrix} c-k, b+n-1; & y \\ 1+b+n+s & ; \end{matrix} \right] \\ & = \frac{b}{m+1} {}_2F_1 \left[\begin{matrix} k, b+n; & x \\ b & ; \end{matrix} \right] {}_2F_1 \left[\begin{matrix} c-k, b+n-1; & y \\ b+n & ; \end{matrix} \right] \\ & + \frac{b(1-b-n)}{(m+1)(b+m+n)} {}_2F_1 \left[\begin{matrix} k, b+n; & x \\ b & ; \end{matrix} \right] {}_2F_1 \left[\begin{matrix} c-k, b+m+n; & y \\ 1+b+m+n & ; \end{matrix} \right] \quad (4.9) \end{aligned}$$

When $P = E = 1$, $G = D = Q = H = 0$ and $p_1 = p_2 + p_3$ in (2.2) and using a reduction formula of Karlsson [5,p. 40(4.5)], we get

$$\begin{aligned} & \sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} \frac{(-1)^{n+r+s} (a)_r (c-b)_s}{(a-b+1)_{r-n} (c)_{n+s}} H_c^{(4)} [a+r, p_2, e, p_3; c+n+s; x, y, y, x] \\ & = \frac{(b)_{m+n}}{(c)_{n+m}} F_{2:2;1}^{2:1;0} \left[\begin{matrix} p_2 + p_3, b+m+n : a, b+n; e & ; x, y \\ b+n, c+m+n & : b & ; -; \end{matrix} \right] \quad (4.10) \end{aligned}$$

In all equations the values of a and b are adjusted in such a way that $(1-b)$, $(1+a-b)$ are not integers; c is neither zero nor a negative integer; Pochhammer symbols and Gamma functions are well defined and denominator parameters in hypergeometric notations are neither zero nor negative integers.

References

- [1] Carlson, B.C.; The need for a new classification of double hypergeometric series, *Proc. Amer. Math. Soc.*, 56(1976), 221–224.
- [2] Erdélyi,A.; Hypergeometric functions of two variables, *Acta. Math.*, 83(1)(1950), 131–164.
- [3] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G.; *Higher Transcendental Functions*, Vol.I (Bateman Manuscript Project), McGraw-Hill Book Co. Inc., New York, Toronto and London, 1953.

- [4] Karlsson, Per W.; Reduction of hypergeometric functions with integral parameters differences, *Nederl. Akad. Wetensch. Proc., Ser.A, 77=Indag.Math.*, 36(1974), 195–198.
- [5] Karlsson, Per W.; A class of hypergeometric functions, *Nederl. Akad. Wetensch. Proc., Ser.A, 79=Indag.Math.*, 38(1976), 36–40.
- [6] Pathan, M.A.; On a summation of ${}_2F_1$, *Math. Chronicle*, 7, (1978), 57–62.
- [7] Rainville, E.D.; *Special Functions*, The Macmillan , New York, 1960, Reprinted by Chelsea Publishing Company, Bronx, New York 1971.
- [8] Srivastava, H.M. and Panda, R.; An integral representation for the product of two Jacobi polynomials, *J. London Math.Soc.*, (2), 12(1976), 419–425.