e-Supplement Submodules and e-Supplemented Modules

Inaam, M.A.Hadi

Department of Mathematics,
College of Education for Pure Science, University of Baghdad

Hassan K. Marhoon

Habibullah Middle School ,
Directorate- General for Education Rusafa,\1
Ministry of Education

Abstract

In this paper , we introduce the concepts e-supplement submodules and e-supplemented modules .We study these concepts and give some basic properties about them.

Key words:

Essential submodule , small submodule , e-small submodule, closed submodule, supplement submodule, supplemented module ,e- supplemented module , e-supplement submodule .

1. Introduction

Let R be a commutative ring with $1 \neq 0$ and M is a unitary R-module . A submodule N of M is called essential (denoted by $N \leq_e M$) , if for any nonzero submodule K of M , $K \cap N \neq (0)$, [2]. And a proper submodule N of M is called small submodule (denoted by $N \ll M$), if $N+K \neq M$, for any proper submodule K of M ,[2]. Recall that a submodule N of an R-module M is called e-small (denoted by $N \ll_e M$), if whenever N+K=M with $K \leq_e M$, then K=M ,[5]. A submodule N is called closed submodule if for each nonzero submodule N of M (denoted by $N \leq_c M$), if N has no proper essential extension submodule in M , that is if $N \leq_e K \leq M$, then N=K [2]. The concept of supplement submodule appeared in [4], where a submodule N of an R-module M is called a supplement submodule in M if M=N+K for some submodule K of M and N is a minimal submodule with this property M=N+K. Equivalently, N is a supplement submodule if M=N+K for some $K \leq M$ and $N \cap K \ll N$.

We introduce and study in section two the concept of e-supplement submodule ,where an essential submodule N of R-module M is called an e-supplement submodule in M, if N+K=M and N is a minimal essential in M with this property . In section 3 we introduce e-supplemented module , if every submodule of M has an e-supplement submodule .

2. e-Supplemented Submodules.

In this section we present a new concept namely e-supplement submodule. We study this concept and give some of its basic properties.

Definition 2.1:

Let $U \leq M$. An essential submodule V of M is called e-supplement of U if U+V=M and V is a minimal essential in M with this property.

The following is a characterization of e-supplement submodule.

Theorem 2.2 :

Let $V \leq_e M$, V is e-supplement of U if and only if M=U+V and $U \cap V \ll_e V$.

Proof: ($\Longrightarrow$)

Let $V \leq_e M$. If V is e-supplement of U, so U+V=M . To prove $U \cap V \ll_e V$.

Assume $(U \cap V)+K=V$, for some $K \leq_e V$. Then $M=U+(U \cap V)+K$, hence M=U+K.But $K \leq_e V$ and $V \leq_e M$, so $K \leq_e M$. On the other hand , $K \leq V$ and V is a minimal essential in M with the property U+V=M. Thus K=V and hence $U \cap V \ll_e V$

($\Longleftarrow$) suppose M=U+V and $U \cap V \ll_e V$. To prove V is e-supplement of U . let $K \leq_e M$ and $K \leq V$ such that M=U+K , we must prove K=V. Since $K \leq_e M$ and $K \leq V$, then $K \leq_e V$.

But $V=M \cap V=(U+K) \cap V=K+(U \cap V)$ by modular law . As $(U \cap V) \ll_e V$ and $K \leq_e V$, imply that K=V and hence V is e-supplement of U.

Remarks and Examples 2.3 :

1- A supplement submodule need not be e-supplement , for example : $< \bar{3} >$ is a supplement of $< \bar{2} >$ in the Z-module Z_6 but $< \bar{3} >$ is not e-supplement , since $< \bar{3} > \nleq_e Z_6$.

2- e- supplement submodule need not be supplement, for example: $<\bar{2}>\leq_e Z_4$, $<\bar{2}>$ is an e-supplement Z_4, but $<\bar{2}>$ is not supplement submodule of Z_4 .
3- Z_{P^∞} is a supplement of N (for each N≤ Z_{P^∞}) . Also it is e-supplement
4- Z_4 is a supplement of any N<Z_4 . Also it is e-supplement

Proposition 2.4 :

Let A,N,K be submodules of an R-module M such that N is e-supplement of M and A is e-supplement of K in M, then A is e-supplement of N in M.

Proof :

Since N is e-supplement of A , then $N\leq_e M$ and A+N=M , $A\cap N\ll_e N$ and since A is e-supplement of K in M , then $A\leq_e M$, and A+K=M , A is minimal essential with the property A+K=M.

To prove A is e-supplement of N . Since $A\leq_e M$, A+N=M , so it is enough to show that A is minimal essential in M with the property A+N=M . Let $L\leq_e M$ and L≤A such that L+N=M . To prove L=A . Since A=M∩A=(L+N)∩A=L+(N∩A) by modular law, then M=L+(N∩A+K=(N∩A)+(L+K). But $(N\cap A)\ll_e N$, then $N\cap A\ll_e M$, also $L\leq_e M$, implies $L+K\leq_e M$. Hence M=L+K. But A is e-supplement of K and $L\leq_e M$, L≤A, so that L=A . Thus A is e-supplement of N ■

Proposition 2.5 :

Let A , N be submodules of an R-module M such that N≤A . If N is e-supplement in M , then N is e-supplement of A

Proof:

Since N is e-supplement in M , then $N\leq_e M$ and there exists K≤M such that N+K=M , $N\cap K\ll_e N$……(1) . Now A=M∩A=(K+N)∩A=N+(K∩A) by modular law . Since $N\cap K\cap A\leq N\cap K\ll_e N$, so $N\cap K\cap A\ll_e N$. Also $N\leq_e A$. Thus N is e-supplement of K∩A in A .

Proposition 2.6 :

Let M be an R-module , let A≤N≤M with N is an e-supplement in M, then A is an e-supplement in N if and only if A is an supplement in M.

Proof : (⟹)

Since N is an e-supplement in M , then $N\leq_e M$ and there exists K≤M such that N+K=M and N is minimal essential with this property . To prove that A is an e-supplement in M . As A is an e-supplement in N , so $A\leq_e N$ and there exists L≤N such that A+L=N and A is minimal essential submodule of N with this property . It follows that M=N+K=A+(L+K).

Since $A\leq_e N$ and $N\leq_e M$, we get $A\leq_e M$. Let $B\leq_e M$, B≤A such that B+L=N , so B+L+K=M . But $B\leq_e M$, then $B+L\leq_e M$. Also B+L≤N and since N is minimal essential such that N+K=M , so that B+L=N , but B≤A and A is a minimal essential submodule such that A+L=N , so B=A and A is minimal essential with property A+(L+K)=M ; i.e. A is an e-supplement of L+K .

(⟸) It follows by Proposition 2.5 .

Proposition 2.7 :

Let M_1 , M_2 be R-module , $M=M_1\oplus M_2$. If A is an e-supplement of K_1 in M_1 , B is an e-supplement of K_2 in M_2 . Then A⊕B is an e-supplement of $K_1\oplus K_2$ in $M_1\oplus M_2$.

Proof :

A is an e-supplement of K_1 in M_1 , then $A\leq_e M$ with $A+K_1=M_1$ and $A\cap K_1\ll_e A$. B is an e-supplement of K_2 in M_2 , then $B\leq_e M_2$ with $B+K_2=M_2$ and $B\cap K_2\ll_e B$.

Now, $M_1\oplus M_2=(A+K_1)\oplus(B+K_2)=(A+B)\oplus(K_1+K_2)$.

Also $(A+B)\cap(K_1+K_2)=(A\cap K_1)\oplus(B\cap K_2)\ll_e A\oplus B$ [5 , Proposition 2.5 (3)].

But $A\leq_e M_1$ and $B\leq_e M_2$, imply $A\oplus B\leq_e M_1\oplus M_2$ [2, Proposition 1.3] . Thus A⊕B is an e-supplement of $K_1\oplus K_2$.

Proposition 2.8 :

Let M be an R-module , if A is an e-supplement of K≤M , let N≤A and N is closed in M , then $\frac{A}{N}$ is an e-supplement in $\frac{M}{N}$

Proof :

A is an e-supplement of K in M, so $A \leq_e M$ and A+K=M , $A \cap K \ll_e A$. To prove that $\frac{A}{N}$ is an e-supplement in $\frac{M}{N}$. First since N<M and $N \leq A \leq_e M$, then $\frac{A}{N} \leq_e \frac{M}{N}$ by [2, Proposition 1.4, P.18]. Now, A+K=M implies $\frac{A+K}{N} = \frac{M}{N}$, hence $\frac{A}{N} + \frac{K+N}{N} = \frac{M}{N}$.

We claim that $\frac{A}{N} \cap \frac{K+N}{N} \ll_e \frac{A}{N}$. Since $\frac{A}{N} \cap \frac{K+N}{N} = \frac{A \cap (K+N)}{N} = \frac{N+(A \cap K)}{N}$ by modules law . Thus $\frac{A}{N} \cap \frac{K+N}{N} = \frac{N+(A \cap K)}{N}$. Let $\frac{L}{N} \leq_e \frac{A}{N}$ such that $\frac{N+(A \cap K)}{N} + \frac{L}{N} = \frac{A}{N}$, then $N+(A \cap K)+L=A$ and hence $(A \cap K)+L=A$ .But $\frac{L}{N} \leq_e \frac{A}{N}$, implies $L \leq_e A$ and since $A \cap K \ll_e A$, then L=A ; that is $\frac{L}{N} = \frac{A}{N}$. It follows that $\frac{A}{N} \cap \frac{K+N}{N} \ll_e \frac{A}{N}$, so $\frac{A}{N}$ is e-supplement of $\frac{K+N}{N}$.

Remark 2.9 :

If A is an e-supplement of B and B is an e-supplement of C , then it is not necessarily that A is an e-supplement of C . For example , let $V = <\bar{2}> \leq Z_4$. V is an e-supplement of Z_4 and Z_4 is an e-supplement of $<\bar{0}>$. But V is not e-supplement of $<\bar{0}>$.

Recall that an R-module is called a multiplication module if for every submodule N of M , there exists an ideal I of R such that IM=N. Equivalently, M is a multiplication module if for every submodule N of M , $N=(N:_R M)M$. [1]

To prove the next result , we prove first the following lemma :

Lemma 2.10 :

Let M be a finitely generated faithful multiplication R-module and let $I \leq J \leq R$. If $I \ll_e J$, then $IM \ll_e JM$.

Proof :

Let $K \leq_e JM$. As $K \leq M$, K=LM for some $L \leq R$, since M is a multiplication R-module . Assume that IM+K=JM , so IM+LM=JM . But M is a finitely faithful multiplication R-module, so I+L=J . But we can show that $L \leq_e J$ as follows , suppose $T \leq J$ and $T \cap L=(0)$. Then $(T \cap L)M=(0)$ and hence $TM \cap LM=(0)$. But $K=LM \leq_e JM$ and $TM \leq JM$, so that TM=(0) and hence T=(0) which implies $L \leq_e J$. But $I \ll_e J$, so L=J . Thus K=LM=JM and $IM \ll_e JM$.

Proposition 2.11:

Let M be a finitely generated faithful multiplication R-module and let $N \leq M$. Then N is an e-supplement in M if and only if [N:M] is e-supplement in R.

Proof : ($\Longrightarrow$)

If N is an e-supplement in M, so $N \leq_e M$ and there exists $K \leq M$ such that N+K=M and $N \cap K \ll_e N$. Since $N \leq_e M$ and M is finitely generated faithful multiplication R-module, then $[N:M] \leq_e R$, also N+K=M , implies [N:M] + [K:M]=R. To prove $[N:M] \cap [K:M] \ll_e [N:M]$. First $[N:M] \cap [K:M]=[N \cap K:M]$. Let $I \leq_e [N:M]$.If $[N \cap K:M]+I=[N:M]$, then $[N \cap K:M]M+IM=[N:M]M$, $[N \cap K]+IM=N$. But $I \leq_e [N:M]$, then $IM \leq_e N$ {by Lemma 2.10} and since $N \cap K \ll_e N$, so IM=N=[N:M]M . As M is a finitely generated faithful multiplication, we get I=[N:M].

Thus $[N \cap K:M] \ll_e [N:M]$.

($\Longleftarrow$) If [N:M] is an e-supplement in R , then $[N:M] \leq_e R$ and there exists $J \leq R$ such that [N:M]+J=R , $[N:M] \cap J \ll_e [N:M]$. Then N+JM=M. But $[N:M] \leq_e R$ implies $N \leq_e M$[1,Th. 2.13]

$N \cap JM=[N:M]M \cap JM=([N:M] \cap J)M$. Also $[N:M] \cap J \ll_e [N:M]$ implies $([N:M] \cap J)M \ll_e [N:M]M$ (by Lemma 2.10) , so that $(N \cap JM) \ll_e N$.

Thus N is an e-supplement in M.

3. e-Supplemented Modules:

In this section, we introduce a new class of module namely e-supplemented module , by using the concept e-supplement submodule. This class of modules is a generalization of the class of supplemented modules.

Definition 3.1:

M is called e-supplement R-module if every submodule of M has an e-supplement submodule .

Example 3.2:

1- Consider Z_4 as Z-module ,
$<\bar{0}>$ has an e-supplement in Z_4 which is Z_4 , $<\bar{2}>$ has an e-supplement in Z_4 which is Z_4,
Z_4 has an e-supplement $<\bar{2}>$. Thus Z_4 is an e-supplemented module

2- Consider Z_6 as Z-module , since each sumodule of Z_6 has an e-supplement submodule which is Z_6 . Thus Z_6 is an e-supplemented module.

3- Consider the Z-module Z , $<\bar{0}>$ has an e-supplement Z . Let N< Z , then N=nZ for some n∈Z, n>1 . Suppose mZ is an e-supplement of nZ , then mZ+nZ=Z . Thus g.c.d (m,n)=1 , so nZ + m'Z=Z and m'Z ⫋ mZ=Z . If m≠±1 , then g.c.d $(n,m^2)=1$, so nZ + $m^2Z=Z$ and $m^2Z⫋mZ$, so every proper submodule of Z has no e-supplement . Thus Z as Z-module is not e-supplemented Z-module .

Remark 3.3:

Let M be a semisimple R-module . Then M is e-supplement

Proof:

Since M is semisimple , then M is the only essential submodule of M and for each N≤M , N+M=M and $N\cap M=N\ll_e M$; that is M is an e-supplement submodule of each submodule N of M

Definition 3.4:

1- Let N be a submodule of a module M . N is called e-weakly supplement of A in M if $N\le_e M$, N+A=M and $N\cap A\ll_e M$. M is called an e-weakly supplemented if every submodule of M has an e-weakly supplement.
2- M is called an e-amply supplement module if for any two submodule X and Y of M with M=X+Y, Y contains an e-supplement of X in M .

Remark 3.5:

For an R- module M. It is clear that

1- M is an amply e-supplemented module implies M is e-supplemented .

Proposition 3.6:

For an R-module M such that $Rad_e M=(0)$. The following statements are equivalent :

1- M is a semisimple module .
2- M is an e-supplement module .
3- M is an e-weakly supplement module.

Proof :

(1) → (2) Since M is semisimple , then M the only essential in M and M+N=M , M∩N=N for any N≤M . But we can show that $N\ll_e N$ as follows. Let N+U=N U≤N. But N is semisimple , then the only essential in N is N it is self , so U=N and hence $N\ll_e N$.

(2) → (3) It is clear ..

(3) → (1) Let N≤M , so there exists K≤M , K is e-weakly supplement , then N+K=M and $N\cap K\ll_e M$. But $Rad_e M=(0)$, then N∩K=(0) . Thus $N\le^{\oplus}K$.

Proposition 3.7 :

Let M , M□ be R-modules and f:M ⟶ M□ be an R-epimophism . If M is an e-amply supplemented module (e-supplemented or e-weakly supplemented) module then so is in M□ .

Proof :

If M is an e-amply supplemented module . Let X,Y ≤M□ , such that M□ =X+Y. Then $f^{-1}(X)+f^{-1}(Y)=M$ (sice f is onto) . But M is an e-amply supplemented module , then $f^{-1}(Y)$ contains C of $f^{-1}(X)$ in M , if $f^{-1}(X)+C=M$ and $f^{-1}(X)\cap C\ll_e C$. Now it is clear that X+ $f^{-1}(C)$=M□. We claim that X∩ $f^{-1}(C)\ll_e f(C)$, where f(C) ≤Y . Since $f^{-1}(X)\cap C\ll_e C$, then $f(f^{-1}(X)\cap C)\ll_e f(C)$. Hence $X\cap f(C)\ll_e f(C)$. Let y∈X∩f(C) , then y=f(C) , for some c∈C , y=f(c)∈X , then $X\cap f(c)\le f(f^{-1}(X)\cap C)\ll_e f(C)$, then $X\cap f(c)\ll_e f(c)$. Thus f(c) is an e-supplement of X in M□ .

The proof in similarly for M is e-supplemented module and M is an e-weakly supplementes module.

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