Čech $\psi\alpha g$-Closed sets in Closure Spaces

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ABSTRACT-*The purpose of this paper is to define and study $\psi\alpha g$-closed sets in closure spaces. We also introduce the concept of $\psi\alpha g$-continuous functions and investigate their properties.*

Keywords: *$\psi\alpha g$-closed sets, $\psi\alpha g$-continuous maps, $\psi\alpha g$-closed maps, $T_{\psi\alpha g}$-spaces.*

1. INTRODUCTION

Čech closure spaces were introduced by E. Čech [3] and then studied by many authors [4][5][7][8]. The concept of generalized closed sets and generalized continuous maps of topological spaces were extended to closure spaces in [2]. A generalization of $\psi\alpha g$-closed sets and $\psi\alpha g$-continuous functions were introduced by the same author [10].

In this paper, we introduce and study the notion of $\psi\alpha g$-closed sets in closure spaces. We define a new class of space namely $T_{\psi\alpha g}$-space and their properties are studied. Further, we introduce a class of $\psi\alpha g$-continuous maps, $\psi\alpha g$-closed maps and their characterizations are obtained.

2. PRELIMIERIES

A map $K: P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a closure operator on X and the pair (X, K) is called a closure space if the following axioms are satisfied
1.$K(\phi) = \phi$
2.$A \subseteq K(A)$ for every $A \subseteq X$
3.$K(A \cup B) = K(A) \cup K(B)$ for all $A, B \subseteq X$.
A closure operator K on a set X is called idempodent if $K(A) = K[K(A)]$ for all $A \subseteq X$.

Definition: 2.1 A subset A of a Čech closure space (X, K) is said to be
(i) Čech closed if $K(A) = A$
(ii) Čech open if $K(X - A) = X - A$
(iii) Čech semi-open if $A \subseteq Kint(A)$
(iv) Čech pre-open if $A \subseteq int[K(A)]$

Definition: 2.2 A Čech closure space (Y, I) is said to be a subspace of (X, K) if $Y \subseteq X$ and $K(A) = K(A) \cap Y$ for each subset, $A \subseteq Y$. If Y is closed in (X, K) then the subspace (Y, I) of (X, K) is said to be closed too.

Definition: 2.3 Let (X, K) and (Y, I) be Čech closure spaces. A map $f: (X, K) \to (Y, K)$ is said to be continuous, if $f(KA) \subseteq Kf(A)$ for every subset $A \subseteq F$.

Definition: 2.4 Let (X, K) and (Y, I) be Čech closure spaces. A map $f: (X, K) \to (Y, I)$ is said to be closed (resp. open), if $f(F)$ is closed (resp. open) subset of (Y, I) whenever F is a closed (resp. open) subset of (X, K).

Definition: 2.5 The product of a family $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ of closure spaces denoted by $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$ is the closure space $(\prod_{\alpha\in I}(X_\alpha, K_\alpha))$ where $\prod_{\alpha\in I} X_\alpha$ denotes the Cartesian product of sets X_α, $\alpha \in I$ and K is Čech closure operator generated by the projections $\pi_\alpha: \prod_{\alpha\in I}(X_\alpha, K_\alpha) \to (X_\alpha, K_\alpha); \alpha \in I$. That is defined by $K(A) = \prod_{\alpha\in I} K_\alpha\, \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha\in I} X_\alpha$).

Clearly, if $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta: \prod_{\alpha\in I}(X_\alpha, K_\alpha) \to (X_\beta, K_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition: 2.6 Let $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and $F \subseteq X_\beta$. Then F is a closed subset of (X_β, K_β) if and only if $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is closed subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$.

Proposition: 2.7 Let $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and $G \subseteq X_\beta$. Then G is a open subset of (X_β, K_β) if and only if $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$.

3. Čech $\psi\alpha g$-Closed sets

Definition: 3.1 Let (X, K) be a Čech closure spaces. A subset $A \subseteq X$ is called a $\psi\alpha g$-closed set if $K(A) \subseteq G$ whenever G is a αg-open subset of (X, K) with $A \subseteq G$. A subset A of X is called a $\psi\alpha g$-open set if its complement is a $\psi\alpha g$-closed subset of (X, K).

Proposition: 3.2 Every closed set is $\psi\alpha g$-closed.

Proof: Let G be a αg-open subset of (X, K) such that $A \subseteq G$. Since A is a closed set, we have $K(A) = A \subseteq G$. Therefore A is $\psi\alpha g$-closed.
The converse need not true as seen in the following example.

Example: 3.3 Let $X = \{a, b, c\}$ and define a closure K on X by $K(\phi) = \phi$, $K\{a\} = \{a\}$; $K\{b\} = \{b, c\}$; $K\{c\} = K\{a, c\} = \{a, c\}$; $K\{a, b\} = K\{b, c\} = KX = X$, Then $\{a, b\}$ is $\psi\alpha g$-closed but it is not closed.

Proposition: 3.4 Let (X, K) be a Čech closure space. If A and B are $\psi\alpha g$-closed subset of (X, K), then $A \cup B$ is also $\psi\alpha g$-closed.

Proof: Let G be a αg-open subset of (X, K) such that $A \cup B \subseteq G$, then $A \subseteq G$ and $B \subseteq G$. Since A and B are $\psi\alpha g$-closed, we have $K(A) \subseteq G$ and $K(B) \subseteq G$. Consequently, $K(A \cup B) = K(A) \cup K(B) \subseteq G$. Therefore $A \cup B$ is $\psi\alpha g$-closed.

Remark: 3.5 The intersection of two $\psi\alpha g$-closed sets need not be $\psi\alpha g$-closed as can be seen by the following example.

Example: 3.6 Let $X = \{a, b, c\}$ and define a closure K on X by $K(\phi) = \phi$, $K\{a\} = \{a, b\}$; $K\{b\} = K\{c\} = K\{b, c\} = \{b, c\}$; $K\{a, b\} = K\{a, c\} = KX = X$ if A=$\{a, b\}$ and $B = \{a, c\}$, then $\{a, b\} \cap \{a, c\} = \{a\}$ which is not $\psi\alpha g$-closed.

Proposition: 3.7 Let (X, K) be a Čech closure spaces. If A is $\psi\alpha g$-closed and F is αg-closed in (X, K), then $A \cap F$ is $\psi\alpha g$-closed.

Proof: Let G be a αg-open subset of (X, K) such that $A \cap F \subseteq G$, then $A \subseteq G \cup (X - F)$ and so, since A is $\psi\alpha g$-closed, $K(A) \subseteq G \cup (X - F)$, then $K(A) \cap F \subseteq G$, since F is αg-closed, $K(A \cap F) \subseteq G$. Therefore $A \cap F$ is $\psi\alpha g$-closed.

Proposition: 3.8 Let (Y, I) be a closed subspace of (X, K). If F is $\psi\alpha g$-closed subset of (Y, I), then F is $\psi\alpha g$-closed subset of (X, K).

Proof: Let G be a αg-open set of (X, K) such that $F \subseteq G$. Since F is $\psi\alpha g$-closed and $G \cap F$ is αg-open $K(F) \cap Y \subseteq G$, but Y is closed subset of (X, K) and $K(F) \subseteq G$, where G is a αg-open set. Therefore F is $\psi\alpha g$-closed set of (X, K).
The following statement is obvious.

Proposition: 3.9 Let (X, K) be a Čech closure space and let $A \subseteq X$. If A is both αg-open and $\psi\alpha g$-closed then A is closed.

Proposition: 3.10 Let (X, K) be a Čech closure space and let K be idempodent. If A is a $\psi\alpha g$-closed subset of (X, K) such that $\subseteq B \subseteq K(A)$, then B is a $\psi\alpha g$-closed subset of (X, K).

Proof: Let G be a αg-open set of (X, K) such that $B \subseteq G$. Then $A \subseteq G$, since A is $\psi\alpha g$-closed, $K(A) \subseteq G$ as G is idempodent, $K(B) \subseteq K(K(A)) = K(A) \subseteq G$. Hence B is $\psi\alpha g$-closed.

Proposition: 3.11 Let (X, K) be a Čech closure space and let $A \subseteq X$. If A is $\psi\alpha g$-closed, then $K(A) - A$ has no non empty αg-closed subset.

Proof: Suppose that A is $\psi\alpha g$-closed. Let F be a αg-closed set of $K(A) - A$. Then $F \subseteq K(A) \cap (X - A)$, so $A \subseteq (X - F)$. Consequently, since A is $\psi\alpha g$-closed $F \subseteq X - K(A)$. Since $F \subseteq K(A)$,$F \subseteq (X - K(A)) \cap K(A) = \phi$. Thus $F = \phi$. Therefore $K(A) - A$ contains no non empty αg-closed subset.

Proposition: 3.12 Let (X, K) be a Čech closure space. A set $A \subseteq X$ is $\psi\alpha g$-open if and only if $F \subseteq X - K(X - A)$ whenever F is αg-closed subset of (X, K) with $F \subseteq A$.

Proof: Suppose that A is $\psi\alpha g$-open and F be a αg-closed subset of (X, K) such that $F \subseteq A$. Then $(X - A) \subseteq (X - F)$. But $X - A$ is $\psi\alpha g$-closed and $X - F$ is αg-open. It follows that $K(X - A) \subseteq X - F$. Ie., $F \subseteq X - K(X - A)$.

Conversely, Let G be a αg-open subset of (X, K) such that $X - A \subseteq G$. Then $X - G \subseteq A$. Therefore $X - U \subseteq K(X - A)$. Consequently, $K(X - A) \subseteq G$. Hence $X - A$ is $\psi\alpha g$-closed and so A is $\psi\alpha g$-open.

Remark: 3.13 The union of two $\psi\alpha g$-open sets need not be $\psi\alpha g$-open.

Proposition: 3.14 Let (X, K) be a Čech closure space. If A is $\psi\alpha g$-open and B is αg-open in (X, K), then $A \cup B$ is $\psi\alpha g$-open.

Proof: Let F be a αg-closed subset of (X, K) such that $F \subseteq A \cup B$. Then $X - (A \cup B) \subseteq X - F$. Hence $(X - A) \cap (X - B) \subseteq X - F$. By proposition 3.7, we have, $(X - A) \cap (X - B)$ is $\psi\alpha g$-closed. Therefore $K[(X - A) \cap (X - B)] \subseteq X - F$. Consequently, $F \subseteq X - K[(X - A) \cap (X - B)] = X - K[X - \cap (A \cup B)]$. Since $F \subseteq X - K[(X - A)]$, then A is $\psi\alpha g$-open. Therefore $A \cup B$ is $\psi\alpha g$-open.

Proposition: 3.15 Let (X, K) be a Čech closure space. If A and B are $\psi\alpha g$-open of (X, K) then $A \cap B$ is $\psi\alpha g$-open.

Proof: Let F be a αg-closed subset of (X, K) such that $F \subseteq A \cap B$. Then $X - (A \cap B) \subseteq X - F$. Hence $(X - A) \cup (X - B) \subseteq X - F$. By proposition 3.4, we have, $(X - A) \cup (X - B)$ is $\psi\alpha g$-closed. Therefore $K[(X - A) \cup (X - B)] \subseteq X - F$. Consequently, $F \subseteq X - K[(X - A) \cup (X - B)] \subseteq X - [X - (A \cap B)]$. By proposition -----, $A \cap B$ is $\psi\alpha g$-open.

Proposition: 3.16 Let (X, K) be a Čech closure space. If A is $\psi\alpha g$-open subsets of (X, K), then $X = G$ whenever G is αg-open and $\big(X - K(X - A)\big) \cup (X - A) \subseteq G$.

Proof: Suppose that A is $\psi\alpha g$-open. Let G be an αg-open subset of (X, K) such that $\big(X - K(X - A)\big) \cup (X - A) \subseteq G$. Then $X - G \subseteq X - \big[\big(X - K(X - A)\big) \cup (X - A)\big]$. Therefore $X - G \subseteq K(X - A) \cap A$ orequivalently, $X - G \subseteq K(X - A) \cap (X - A)$. But $X - G$ is αg-closed and $X - A$ is $\psi\alpha g$-closed. Then by proposition 3.11, $X - G = \phi$. Consequently, $X = G$.

Proposition: 3.17 Let (X, K) be a Čech closure space and let $A \subseteq X$. If A is $\psi\alpha g$-closed, then $K(A) - A$ is $\psi\alpha g$-open.

Proof: Suppose that A is $\psi\alpha g$-closed. Let F be a αg-closed set of (X, K) such that $F \subseteq K(A) - A$, By proposition 3.11, $K(A) - A = \phi$, and hence $F \subseteq X - [(X - K(X - A))]$. By proposition 3.12 $K(A) - A$ is $\psi\alpha g$-open.

Proposition: 3.18 Let $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ be a family of closure spaces. Let $\beta \in I$ and $G \subseteq X_\beta$. Then G is a $\psi\alpha g$-open subset of (X_β, K_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a $\psi\alpha g$-open subset of $\prod_{\alpha \in I}(X_\alpha, K_\alpha)$.

Proof: Let F be a $\psi\alpha g$-closed subset of (X_α, K_α) such that $F \subseteq G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is αg-closed and G is $\psi\alpha g$-open in (X_β, K_β), $\pi_\beta(F) \subseteq X_\beta - K_\beta(X_\beta - G)$. Therefore $F \subseteq \pi_\beta^{-1}(X_\beta - K_\beta(X_\beta - G)) = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} K_\alpha \pi_\alpha \left(\prod_{\alpha \in I} X_\alpha - G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \right)$. Hence $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a $\psi\alpha g$-closed subset of $\prod_{\alpha \in I}(X_\alpha, K_\alpha)$.

Conversely, Let F be a αg-closed subset of (X_β, K_β) such that $F \subseteq G$. Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. Since $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is αg-closed and $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is $\psi\alpha g$-open in $\prod_{\alpha \in I}(X_\alpha, K_\alpha)$. $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subseteq \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} K_\alpha \pi_\alpha \left(\prod_{\alpha \in I} X_\alpha - G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \right)$. By proposition 3.12 Therefore $\prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} K_\alpha \pi_\alpha (X_\beta - G) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subseteq \prod_{\alpha \in I} X_\alpha - F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha = (X_\beta - F) \prod_{\alpha \in I} X_\alpha$.

Consequently, $K_\beta(X_\beta - G) \subseteq X_\beta - F$ implies $F \subseteq X_\beta - K_\beta(X_\beta - G)$. Hence G is a $\psi\alpha g$-open subset of (X_β, K_β).

Proposition: 3.19 Let $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ be a family of closure spaces. Let $\beta \in I$ and $F \subseteq X_\beta$. Then F is $\psi\alpha g$-closed subset of (X_β, K_β) if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a $\psi\alpha g$-closed subset of $\prod_{\alpha \in I}(X_\alpha, K_\alpha)$.

Proof: Let F be a $\psi\alpha g$-closed subset of (X_β, K_β). Then $X_\beta - F$ is an $\psi\alpha g$-open subset of (X_β, K_β). By proposition 3.18, $X_\beta - F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha =$

$\prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a $\psi\alpha g$-open subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$. Hence $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a $\psi\alpha g$-closed subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$.

Conversely, let G be a $\psi\alpha g$-open subset of (X_β, K_β) such that $F \subseteq G$. Then $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$. Since $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is $\psi\alpha g$-closed and $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is αg-open in $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$. $\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} K_\alpha \pi_\alpha (F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\beta) \subseteq G \times \prod_{\alpha\in I} X_\alpha$. Consequently, $K_\beta(F) \subseteq G$. Therefore, F is a $\psi\alpha g$-closed subset of (X_β, K_β).

Proposition: 3.20 Let $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ be a family of closure spaces, for each $\beta \in I$ and let $\pi_\beta: \prod_{\alpha\in I} X_\alpha \to X_\beta$ be a projection map. Then (i) If F is a $\psi\alpha g$-closed subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$, then $\pi_\beta(F)$ is a $\psi\alpha g$-closed subset of (X_β, K_β).

(ii) If F is a $\psi\alpha g$-closed subset of (X_β, K_β), then $\pi_\beta^{-1}(F)$ is a $\psi\alpha g$-closed subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$.

Proof: Let F be a $\psi\alpha g$-closed subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$ and let G be a αg-open subset of (X_β, K_β) such that $\pi_\beta(F) \subseteq G$. Then $F \subseteq \pi_\beta^{-1}(G) = G \times \prod_{\alpha\in I} X_\alpha$. Since F is a $\psi\alpha g$-closed and $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\beta$ is αg-open. $\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} K_\alpha \pi_\alpha \left(G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha \right)$. Consequently, $K_\beta \pi_\beta(F) \subseteq G$. Hence $\pi_\beta(F)$ is a $\psi\alpha g$-closed subset of (X_β, K_β).

(ii) Let F be a $\psi\alpha g$-closed subset of (X_β, K_β). Then $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$. Therefore we have, $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a $\psi\alpha g$-closed subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$. Therefore $\pi_\beta^{-1}(F)$ is a $\psi\alpha g$-closed subset of $\prod_{\alpha\in I}(X_\alpha, K_\alpha)$.

Definition: 3.21A closure space (X, K) is said to be a $T_{\psi\alpha g}$-space if every $\psi\alpha g$-closed subset of (X, K) is closed.

Proposition: 3.22 Let (X, K) be closure space. Then

(i)If (X, K) is a $T_{\psi\alpha g}$-space then every singleton subset of X is either αg-closed or open.

(ii)If every singleton subset of X is a αg-closed subset of (X, K), then (X, K) is a $T_{\psi\alpha g}$-space.

Proof: (i) Suppose that (X, K) is a $T_{\psi\alpha g}$-space. Let $x \in X$ and assume that $\{x\}$ is not αg-closed. Then $X - \{x\}$ is not αg-open. This implies $X - \{x\}$ is $\psi\alpha g$-closed. Since X is the only αg-open set which contains $-\{x\}$. Since (X, K) is a $T_{\psi\alpha g}$-space, $X - \{x\}$ is closed or equivalently $\{x\}$ is open.

(ii) Let A be a $\psi\alpha g$-closed subset of (X, K). Suppose that $x \notin A$. Then $\{x\} \subseteq X - \{x\}$. Since A is $\psi\alpha g$-closed and $X - \{x\}$ is αg-open, $K(A) \subseteq X - \{x\}$. That is $\{x\} \subseteq X - K(A)$.

Hence $x \notin K(A)$ and thus $K(A) \subseteq A$. Therefore A is closed subset of (X, K). Hence (X, K) is a $T_{\psi\alpha g}$-space.

4. Čech $\psi\alpha g$-Continuous maps

Definition: 4.1 Let (X, K) and (Y, l) be a Čech closure space. A mapping $f: (X, K) \to (Y, l)$ is said to be $\psi\alpha g$-continuous, if $f^{-1}(F)$ is $\psi\alpha g$-closed closed set of (X, K) for every closed set F in (Y, l).

Proposition: 4.2 Every continuous map is $\psi\alpha g$-continuous.

Proof: Let $f: (X, K) \to (Y, l)$ be continuous. Let F be a closed set of (Y, l). Since f is continuous, then $f^{-1}(F)$ is closed set of (X, K). Since every closed set is $\psi\alpha g$-closed of (X, K), we have $f^{-1}(F)$ is closed set of (X, K). Therefore f is $\psi\alpha g$-continuous.

Proposition: 4.3 Let (X, K) be a $T_{\psi\alpha g}$-space and (Y, l) be a Čech closure space. If $f: (X, K) \to (Y, l)$ is said to be αg-continuous, then f is $\psi\alpha g$-continuous.

Proof: Let F be a closed set of (Y, l). Since f is αg-continuous, then $f^{-1}(F)$ is αg-closed set of (X, K). Since (X, K) is a $T_{\psi\alpha g}$-space, $f^{-1}(F)$ is a $\psi\alpha g$-closed set of (X, K). Hence f is $\psi\alpha g$-continuous.

The following statement is obvious.

Proposition: 4.4 Let (X, K), (Y, l) and (Z, m) be closure spaces. If $f: (X, K) \to (Y, l)$ is $\psi\alpha g$-continuous and $g: (Y, l) \to (Z, m)$ is continuous, then $g \circ f: (X, K) \to (Z, m)$ is $\psi\alpha g$-continuous.

Proposition: 4.5 Let (Z, m) be closure spaces and let (Y, I) be a $T_{\psi\alpha g}$-space. If $f: (X, K) \to (Y, I)$ and $g: (Y, I) \to (Z, m)$ are $\psi\alpha g$-continuous, then $g \circ f: (X, K) \to (Z, m)$ is $\psi\alpha g$-continuous.

Proof: Let F be a closed set of (Z, m). Since g is $\psi\alpha g$-continuous, then $g^{-1}(F)$ is $\psi\alpha g$-closed set of (Y, I). Since (Y, I) is a $T_{\psi\alpha g}$-space, $g^{-1}(F)$ is a closed set of (Y, I) which implies that $(g \circ f)^{-1}(F)$ is a $\psi\alpha g$-closed subset of (X, K). Hence $g \circ f$ is $\psi\alpha g$-continuous.

Proposition: 4.6 Let $\{(X_\alpha, K_\alpha);\ \alpha \in I\}$ and $\{(Y_\alpha, I_\alpha);\ \alpha \in I\}$ be families of closure spaces. For each $\alpha \in I$, let $f_\alpha: X_\alpha \to Y_\alpha$ be a map and $f: \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be a map defined by $f((X_\alpha)_{\alpha \in I}) = (f_\alpha(X_\alpha)_{\alpha \in I})$. If $f: \prod_{\alpha \in I}(X_\alpha, K_\alpha) \to \prod_{\alpha \in I}(Y_\alpha, I_\alpha)$ is $\psi\alpha g$-continuous, then $f_\alpha: (X_\alpha, K_\alpha) \to (Y_\alpha, I_\alpha)$ is $\psi\alpha g$-continuous for each $\alpha \in I$.

Proof: Let $\beta \in I$ and F be a closed subset of (Y_β, I_β). Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha \in I}(Y_\alpha, I_\alpha)$. Since f is $\psi\alpha g$-continuous, $f^{-1}\left(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha\right) = f_\beta^{-1}(F) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a $\psi\alpha g$-closed subset of $\prod_{\alpha \in I}(X_\alpha, K_\alpha)$. By proposition...... $f_\beta^{-1}(F)$ is a $\psi\alpha g$-closed subset of (X_β, K_β). Hence F_β is $\psi\alpha g$-continuous.

Definition: 4.7 Let (X, K) and (Y, I) be a closure spaces. A map $f: (X, K) \to (Y, I)$ is called $\psi\alpha g$-irresolute if $f^{-1}(F)$ is a $\psi\alpha g$-closed set of (X, K) for every $\psi\alpha g$-closed set F in (Y, I).

Definition: 4.8 Let (X, K) and (Y, I) be a closure spaces. A map $f: (X, K) \to (Y, I)$ is called $\psi\alpha g$-closed if $f(F)$ is a $\psi\alpha g$-closed subset of (Y, I) for every closed set F of (X, K).

Proposition: 4.9 Let (X, K), (Y, I) and (Z, m) be closure spaces. If $f: (X, K) \to (Y, I)$ and $g: (Y, I) \to (Z, m)$ be a map, then

(i)If f is closed and g is $\psi\alpha g$-closed, then $g \circ f$ is $\psi\alpha g$-closed.

(ii) If $g \circ f$ is $\psi\alpha g$-closed and f is $\psi\alpha g$-continuous and surjective, then g is $\psi\alpha g$-closed.

(iii) If $g \circ f$ is closed and g is $\psi\alpha g$-continuous and injective, then f is $\psi\alpha g$-closed.

Proposition: 4.10 A map $f: (X, K) \to (Y, I)$ is $\psi\alpha g$-closed if and only if, for each subset B of Y and each open subset G with $f^{-1}(B) \subseteq G$, there is a $\psi\alpha g$-open subset V of (Y, I) such that $B \subseteq V$ and $f^{-1}(V) \subseteq G$.

Proof: Suppose f is $\psi\alpha g$-closed. Let B be a subset of (Y, I) and G be an open subset of (X, K) such that $f^{-1}(B) \subseteq G$. Then $f(X - G)$ is a $\psi\alpha g$-closed subset of (Y, I). Let $V = Y - f(X - G)$. Since V is $\psi\alpha g$-open and $f^{-1}(V) = f^{-1}\big(Y - f(X - G)\big) = X - f^{-1}\big(f(X - G)\big) \subseteq X - (X - G) = G$. Therefore, V is $\psi\alpha g$-open, $B \subseteq V$ and $f^{-1}(V) \subseteq G$.

Conversely, suppose F is a closed subset of (X, K), then $f^{-1}\big(Y - f(F)\big) \subseteq X - F$ and $X - F$ is open. By hypothesis, there is a $\psi\alpha g$-open subset V of (Y, I) such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore $F \subseteq X - f^{-1}(V)$. Hence $Y - V \subseteq f(F) \subseteq f\big(X - f^{-1}(V)\big) \subseteq Y - V \Rightarrow f(F) = Y - V$. Thus $f(F)$ is $\psi\alpha g$-closed. Therefore f is $\psi\alpha g$-closed.

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