CI-algebras and its Fuzzy Ideals

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ABSTRACT

In this paper we develop the idea of fuzzy ideals in cartesian product of CI-algebras and obtain some new results. Finally we investigate how to extend a given fuzzy ideal of a CI-algebra to that of another CI-algebra.

Keywords: *CI-algebra, Ideals, Fuzzy ideal*

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1. INTRODUCTION

In 1966,Y.Imai and K.Iseki [2] introduced the notion of a BCK-algebra. There exist several generalizations of BCK-algebras,such as BCI-algebras [3],BCH-algebras [1],BH-algebras [4],d-algebras [8],etc. In [5],H.S.Kim and Y.H.kim introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra.As a generalization of Be-algebras,B.L.Meng [7] introduced the notion of CI-algebras and discussed its important properties.The concept of fuzzification of ideals in CI-algebra have introduced by Samy M. Mostafa, Mokthar A. Abdel Naby ,Osama R.Elgendy [10]. In this paper we develop the idea of fuzzy ideals in cartesian product of CI-algebras and obtain some new results. BY establishing that if X is a CI-algebra then F(X), the class of all functions f : $X \rightarrow X$ .is also a CI-algebra, we extend a given fuzzy ideal of x to that of F(X).

2. PRELIMINARIES

Definition 2.1. ([7])- A system $(X; *, 1)$ consisting of a non –empty set X , a binary operation $*$ and a fixed element 1, is called a CI – algebra if the following conditions are satisfied :

1. (CI 1) $x * x = 1$
2. (CI 2) $1 * x = x$
3. (CI 3) $x * (y * z) = y * (x * z)$

for all $x, y, z \in X$

Example 2.2. Let $X = R^+ = \{x \in R : x > o\}$

For $x, y \in X$, we define

$x * y = y \cdot \frac{1}{x}$

Then $(X; *, 1)$ is a CI – algebra

Definition 2.3.([7])A non – empty subset I of a CI – algebra X is called an ideal of X if

(1) $x \in X$ and $a \in I \Rightarrow x * a \in I$;
(2) $x \in X$ and $a, b \in I \Rightarrow (a * (b * x)) * x \in I$.

Lemma 2.4. ([7]) In a CI – algebra following results are true:

(1) $x * ((x * y) * y) = 1$

(2) $(x * y) * 1 = (x * 1) * (y * 1)$
(3) $1 \leq x$ imply $x = 1$

for all $x, y \in X$.

Lemma 2.5. ([6]) - (i) Every ideal I of X contains 1

(ii) If I is an ideal of X then $(a * x) * x \in I$ for all $a \in I$ and $x \in X$

(iii) If I_1 and I_2 are ideals of X then so is $I_1 \cap I_2$.

Theorem 2.6. ([9])- Let $(X; *, 1)$ be a system consisting of a non-empty set X , a binary operation $*$ and a fixed element 1. Let $Y = X \times X$. For $u = (x_1, x_2)$, $v = (y_1, y_2)$ a binary operation ''$\odot$''is defined in Y as

$$u \odot v = (x_1 * y_1, x_2 * y_2)$$

Then $(Y; \odot, (1, 1))$ is a CI- algebra iff $(X; *, 1)$ is a CI -algebra .

Theorem 2.7.([9])- Let A and B be subsets of a CI – algebra X . Then A x B is an ideal of $Y = X \otimes X$ iff A and B are ideals of X .

3. FUZZY IDEALS .

Definition (3.1.) ([10]) - Let $(X; *, 1)$ be a CI – algebra and let μ be a fuzzy set in X . Then μ is called a fuzzy ideal of X if it satisfies the following conditions:

(1) $(\forall x, y \in X) (\mu(x * y) \geq \mu(y))$,

(2) $(\forall x, y, z \in X) (\mu((x * (y * z)) * z) \geq \min \{\mu(x), \mu(y)\})$

Theorem (3.2.) ([10]) - Let μ be a fuzzy set in a CI – algebra $(X; *, 1)$. and let

$U(\mu ; \alpha) = \{x \in X: \mu(x) \geq \alpha\}$ where $\alpha \in [0, 1]$.

Then μ is a fuzzy ideal of X iff $(\forall \alpha \in [0, 1]) (U(\mu ; \alpha) \neq \phi \Rightarrow U(\mu ; \alpha)$ is an ideal of X).

Proposition (3.3) ([10]) - Let μ be a fuzzy ideal of a CI – algebra $(X; *, 1)$. Then

(a) $\mu(1) \geq \mu(x)$ for all $x \in X$;

(b) $\mu((x * y) * y) \geq \mu(x)$ for all $x, y \in X$;

(c) $x, y \in X$ and $x \leq y \Rightarrow \mu(x) \leq \mu(y)$,

i. e., a fuzzy ideal μ is order preserving .

Now we establish some results for fuzzy ideals on Cartesian product of CI– algebras.

Theorem (3.4) - Let μ be a fuzzy set on a CI – algebra X and let $Y = X \times X$. Let μ_1, μ_2, μ_3 be fuzzy sets on Y defined as

$$\mu_1(x, y) = \mu(x)$$

$$\mu_2(x, y) = \mu(y)$$

$\mu_3(x , y) = \min\{\mu(x) , \mu(y)\}$

Then, (a) μ_1 is a fuzzy ideal of Y iff μ is a fuzzy ideal of X;

(b) μ_2 is a fuzzy ideal of Y iff μ is a fuzzy ideal of X;

(c) μ_3 is a fuzzy ideal of Y iff μ is a fuzzy ideal of X.

Proof :- For any real $\alpha \in [0, 1]$, let

$U(\mu ; \alpha) = \{x \in X: \mu(x) \geq \alpha\}$;

$U_1(\mu_1 ; \alpha) = \{(x, y) \in Y : \mu_1(x , y) = \mu(x) \geq \alpha\}$;

$U_2(\mu_2 ; \alpha) = \{(x, y) \in Y : \mu_2(x , y) = \mu(y) \geq \alpha\}$;

and $U_3(\mu_3 ; \alpha) = \{(x, y) \in Y : \mu_3(x , y)) \geq \alpha\}$;

Then we see that $U_1(\mu_1 ; \alpha) = U(\mu ; \alpha) \times X$

$U_2(\mu_2 ; \alpha) = X \times U(\mu ; \alpha)$

$U_3(\mu_3 ; \alpha) = U(\mu ; \alpha) \times U(\mu ; \alpha)$

Now using theorem (2.7) we see that

(i) $U_1(\mu_1 ; \alpha)$ is an ideal in Y iff $U(\mu ; \alpha)$ is an ideal in X
(ii) $U_2(\mu_2 ; \alpha)$ is an ideal in Y iff $U(\mu \ \alpha)$ is an ideal in X
(iii) $U_3(\mu_3 ; \alpha)$ is an ideal in Y iff $U(\mu ; \alpha)$ is an ideal in X.

for all real $\alpha \in [0, 1]$.

Using theorem (3.2) we get the result.

Definition (3.5):- Let μ be a fuzzy set in $Y = X \times X$. Let μ_1 and μ_2 be fuzzy sets defined in X as

$\mu_1(x) = \mu(x, 1)$

and $\mu_2(x , y) = \mu(1, x)$

Theorem (3.6):- μ is a fuzzy ideal of Y iff μ_1 and μ_2 are fuzzy ideals of X.

Proof :- Let μ be a fuzzy ideal of Y . Then

$U(\mu ; \alpha) = \{(x, y) \in Y : \mu(x , y) \geq \alpha\} = A \times B$ (say)

is an ideal in Y . This means that A and B are ideal in X [theorem (2.7)]. So $1 \in A \cap B$.

Now we prove that, $U_1(\mu_1 ; \alpha) = \{x \in X : \mu_1(x) \geq \alpha\} = A$

and $U_2(\mu_2 ; \alpha) = \{x \in X : \mu_2(x) \geq \alpha\} = B$.

We see that , $x \in A, 1 \in B \Leftrightarrow (x, 1) \in U(\mu ; \alpha) \Leftrightarrow \mu(x, 1) \geq \alpha \Leftrightarrow \mu_1(x) \geq \alpha \Leftrightarrow x \in U_1(\mu_1 ; \alpha)$ Hence $A = U_1(\mu_1 ; \alpha)$. Similarly we can prove that $B = U_2(\mu_2 ; \alpha)$.

Thus we see that $U_1(\mu_1 ; \alpha)$ and $U_2(\mu_2 ; \alpha)$ are ideals in X for every $\alpha \in [0, 1]$. Hence μ_1 and μ_2 are fuzzy ideals in X.

Conversely, suppose that μ_1 and μ_2 are fuzzy ideals in X . Then, $U_1(\mu_1 ; \alpha) = \{x \in X : \mu_1(x) \geq \alpha\}$ and $U_2(\mu_2 ; \alpha) = \{x \in X : \mu_2(x) \geq \alpha\}$ are ideals in X for every $\alpha \in [0, 1]$.

So $1 \in U_1(\mu_1 ; \alpha) \cap U_2(\mu_2 ; \alpha)$ which means that $\mu(1, 1) \geq \alpha$ [def.(3.5)]

Now $U(\mu ; \alpha) = \{(x, y) \in Y : \mu (x , y) \geq \alpha\} = A \times B$ (Say) contains (1, 1) .

We see that, $x \in U_1(\mu_1 ; \alpha) \Leftrightarrow \mu_1(x) \geq \alpha \Leftrightarrow \mu(x, 1) \geq \alpha \Leftrightarrow (x, 1) \in U(\mu ; \alpha) \Leftrightarrow x \in A, 1 \in B$
So we have $A = U_1 (\mu_1 ; \alpha)$. Similarly we see that $B = U_2 (\mu_2 ; \alpha)$.

Thus $A \times B = U(\mu ; \alpha)$ is an ideal in Y for every $\alpha \in [0, 1]$ [theorem (2.7)] Hence μ is a fuzzy ideal of Y.

Now we discuss fuzzy ideal for function algebra. For this we have to prove following results.

Theorem (3.7)- Let (X; *, 1) be a CI – algebra and let F(X) be the class of all functions $f : X \to X$. Let a binary operation ‘‘ o ’’ be defined in F(X) as follows:

For $f, g \in F(X)$ and $x \in X$,

$$(f \circ g)(x) = f(x) * g(x).$$

Then $(F(X) ; o , 1^{\sim})$ is a CI – algebra where $1^{\sim}$ is defined as $1^{\sim}(x) = 1$ for all $x \in X$.

Here two functions $f, g \in F(X)$ are equal iff $f(x) = g(x)$ for all $x \in X$.

Proof : Let $f, g, h \in F(X)$. Then for $x \in X$, we have

(i) $(f \circ f)(x) = f(x) * f(x) = 1 = 1^{\sim}(x) \Rightarrow f \circ f = 1^{\sim}$,
(ii) $(1^{\sim} \circ f)(x) = 1^{\sim}(x) \circ f(x) = f(x) \Rightarrow 1^{\sim} \circ f = f$,
(iii) $(f \circ (g \circ h))(x) = f(x) * (g \circ h)(x)$

$$= f(x) * (g(x) * h(x))$$

$$= g(x) * (f(x) * h(x))$$

$$= g(x) * (f \circ h)(x)$$

$$= (g \circ (f \circ h))(x).$$

$$\Rightarrow f \circ (g \circ h) = g \circ (f \circ h).$$

This proves that $(F(X) ; o , 1^{\sim})$ is a CI – algebra.

Theorem (3.8):- Let (X; * , 1) be a CI – algebra and let $(F(X) ;o, 1^{\sim})$ be CI – algebra considered in the above theorem. Then

I is an ideal of X $\Leftrightarrow$ F(I) is an ideal of F(X).

Proof : Let I be an ideal of X. For $f \in F(X)$ and $g \in F(I)$ we have $f(x) \in X$ and $g(x) \in I$ for all $x \in X$.

So $(f \text{ o } g)(x) = f(x) * g(x) \in I$ for all $x \in X$.

This gives $f \text{ o } g \in F(I)$.

Again for $g, h \in F(I)$, $f \in F(X)$ and $x \in X$, we have

$((g \text{ o } (h \text{ o } f)) \text{ o } f)(x) = (g \text{ o } (h \text{ o } f))(x) * f(x),$

$= (g(x) * (h(x) * f(x))) * f(x) \in I.$

So $(g \text{ o } (h \text{ o } f)) \text{ o } f \in F(I)$.

Hence F(I) is an ideal in F(X).

Conversely, suppose that F(I) is an ideal of F(X). Then

$f \in F(X)$ and $g \in F(I) \Rightarrow f \text{ o } g \in F(I)$

$\Rightarrow (f \text{ o } g)(t) \in I$ for all $t \in X$

$\Rightarrow f(t) * g(t) \in I$ for all $t \in X$. (3.1)

Also $f \in F(X)$ and $g, h \in F(I)$

$\Rightarrow (g \text{ o } (h \text{ o } f)) \text{ o } f \in F(I)$

$\Rightarrow ((g \text{ o } (h \text{ o } f)) \text{ o } f)(t) \in I$ for all $t \in X$,

$\Rightarrow (g(t) * (h(t)) * f(t)) * f(t) \in I$ for all $t \in X$. (3.2)

Let $x \in X$ and $a \in I$, We consider function f_x and f_a defined as $f_x(t) = x$ and $f_a(t) = a$ for all $t \in X$. (3.3)

Now $f_x \in F(X)$ and $f_a \in F(I)$. So $f_x \text{ o } f_a \in F(I)$.

This implies that $(f_x \text{ o } f_a)(t) = f_x(t) * f_a(t) = x * a \in I$ for all $t \in X$ [from (3.1)].

Again let $a, b \in I$ and $x \in X$. We consider functions f_a, f_b and f_x as defined by (3.3).

Then f_a, $f_b \in F(I)$ and $f_x \in F(X)$.

So $(f_a \text{ o } (f_b \text{ o } f_x)) \text{ o } f_x \in F(I)$.

This gives $((f_a \text{ o } (f_b \text{ o } f_x)) \text{ o } f_x)(t) = (a * (b * x)) * x \in I$. for all $t \in X$, [from (3.2)].

Hence I is an ideal of X.

Definition (3.9):- Let μ be a fuzzy set defined on a finite CI – algebra (X; * , 1). Let (F(X) ; o , $\tilde{1}$) be the CI – algebra considered in the theorem (3.7). We extend $\bar{\mu}$ on F (X) as

$$\overline{\mu}(f) = \min\{\mu(f(x)) : x \in X\}.$$

We prove the following result.

Lemma(3.10) :- If $\mu(x) \le \mu(1)$ for all $x \in X$ then $\overline{\mu}(f) \le \overline{\mu}(\tilde{1})$ for all $f \in F(X)$.

Proof :- First of all we observe that $\overline{\mu}(\tilde{1}) = \mu(1)$

Since $\overline{\mu}(\tilde{1}) = \min\{\mu(\tilde{1}(x)) : x \in X\}$

$= \mu(1)$.

Now $\overline{\mu}(f) = \min\{\mu(f(x)) : x \in X\}$

$\le \mu(1)$, since $\mu(f(x)) \le \mu(1)$ for all $x \in X$

$= \overline{\mu}(\tilde{1})$.

Lemma(3.11) :- $F(U(\mu ; \alpha)) = U(\overline{\mu} ; \alpha)$ for every $\alpha \in [o, 1]$.

Proof :- First of all we observe that for any $\alpha \in [o, 1]$,

$U(\overline{\mu} ; \alpha) \ne \phi \Leftrightarrow U(\mu ; \alpha) \ne \phi$

Let $U(\overline{\mu} ; \alpha) \ne \phi$ and $f \in U(\overline{\mu} ; \alpha)$

Then $\overline{\mu}(f) \ge \alpha$. So $\min\{\mu(f(x) : x \in X\} \ge \alpha$.

This implies that $\mu(f(x)) \ge \alpha$ for some $x \in X$,

i.e., $f(x) \in U(\mu : \alpha)$, and so $U(\mu ; \alpha) \ne \phi$.

Again let $U(\mu ; \alpha) \ne \phi$ and $a \in U(\mu ; \alpha)$.

Then $\mu(a) \ge \alpha$. If we choose $f_a \in F(X)$ such that $f_a(x) = a$ for all $x \in X$. Then $\overline{\mu}(f_a) = \min\{\mu(f_a(x)) : x \in X\} = \mu(a) \ge \alpha$, i. e, $f_a \in U(\overline{\mu} ; \alpha)$ and so $U(\overline{\mu} ; \alpha) \ne \phi$.

Now we see that , $f \in F(U(\mu; \alpha)) \Leftrightarrow f(x) \in U(\mu; \alpha)$ for all $x \in X$

$\Leftrightarrow \mu(f(x)) \ge \alpha$ for all $x \in X$

$\Leftrightarrow \min\{\mu(f(x)) : x \in X\} \ge \alpha \Leftrightarrow \overline{\mu}(f) \ge \alpha \Leftrightarrow f \in U(\overline{\mu} : \alpha)$

This gives $U(\overline{\mu} ; \alpha) = F(U(\mu; \alpha))$.

Corollary (3.12) :- If $U(\mu; \alpha)$ is an ideal in X then $U(\overline{\mu} ; \alpha)$ is an ideal in F(X).

Proof :- This follows from above lemma and theorem (3.8).

Theorem (3.13):- If μ is a fuzzy ideal of X then so is $\overline{\mu}$ on F(X).

Proof :- Let μ be a fuzzy ideal of X . Then for every $\alpha \in [0, 1]$

$U(\mu; \alpha) \neq \phi \Rightarrow U(\mu; \alpha)$ is an ideal.in X

So $F(U(\mu; \alpha))$ is an ideal in F (X) by theorem (3.8).

Now if $\alpha \in [0, 1]$ and $U(\overline{\mu}; \alpha) \neq \phi$ then from discussion given in lemma (3.11) we see that, $U(\overline{\mu}; \alpha) = F(U(\mu; \alpha))$ and so $U(\overline{\mu}; \alpha)$ is an ideal in F(X).

This proves that $\overline{\mu}$ is a fuzzy ideal in F(X).

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