Fractional order Hirota-Satsuma coupled KdV equation by Homotopy perturbation transforms method

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Abstract: *In this paper, we have used homotopy perturbation method and Laplace transformation to determine approximate solutions which converge to exact solution of generalized Hirota−Satsuma coupled KdV equation. The nonlinear terms handled by the use of He's polynomial.*

Keywords: *Homotopy perturbation method, Laplace transform method, Generalized Hirota-Satsuma Coupled KdV Equation.*

1. Introduction:

A number of methods have been proposed in the literature recently for solving different kinds of physical and mathematical problems. Among them, the homotopy perturbation is the advance approach for finding the approximate analytical solution of linear and nonlinear problems. The method was first proposed by He [1–3], provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems of equations representing real physical problems. The solution of differential equations of fractional order is much involved. Though many exact solutions for linear fractional differential equations have been found but in general, there exists no method that yields an exact solution for nonlinear fractional differential equations. The nonlinear phenomena have important effects on applied mathematics, physics etc. In recent years, many authors have studied the solutions of nonlinear partial differential equations by using homotopy perturbation method, the Variational iteration method [4-6] and the Adomian decomposition method [7].[a]

An elementary introduction to the homtopy perturbation method can be found in [8-15], improved homotopy perturbation method is given in [16-17]. Many researchers [18–21] have obtained the series solution of the fractional differential equations and integral equation by using HPM, In this article, we have applied the homotopy perturbation method coupled with the Laplace transformation method to determine the solution of time fractional Hirota-Satsuma Coupled KdV equation know as homotopy perturbation transform method [22–23].

2. Definition: The Laplace transform of $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt.$$

where s is the parameter it may be real or complex,

3. Definition: The Laplace transform of the Riemann–Liouville fractional integral [24] is defined as

$$L[I^\alpha f(t)] = s^{-\alpha} F(s).$$

4. Definition: The Laplace transform of the Caputo fractional derivative [24] is defined as

$$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} f^{(k)}(0), \qquad n-1 < \alpha \le n.$$

5. Homotopy perturbation transforms method (HPTM)

We consider the following nonlinear fractional differential equation:

$$\begin{aligned} &D_t^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = 0, \quad t>0, \; x \in R, \; 0<\alpha \le 1, \\ &u(x,0) = h(x), \end{aligned} \tag{1}$$

where $D^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative operator, R is the linear operator, N is the nonlinear operator.

Taking the Laplace transform on both sides of Eq. (1), we have

$$L[D^\alpha u(x,t)] + L[Ru(x,t) + Nu(x,t)] = 0 \tag{2}$$

The Laplace transform of the Caputo fractional derivative is defined as

$$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} f^{(k)}(0), \qquad n-1 < \alpha \le n. \tag{3}$$

Now, using the property of Laplace for fractional derivative as given by equation (3), we obtain

$$L[u(x,t)] = \frac{1}{s} h(x) - \frac{1}{s^\alpha} L[Ru(x,t) + Nu(x,t)], \tag{4}$$

On operating the inverse Laplace transform on both sides of equation (4), we get

$$u(x,t) = h(x) - L^{-1}\left(\frac{1}{s^\alpha} L[Ru(x,t) + Nu(x,t)]\right). \tag{5}$$

Further,applying the homotopy perturbation technique; the solution can be expressed as a power series in terms of p i.e.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{6}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). The nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{7}$$

where H_n are He's polynomials of $u_0, u_1, u_2, ..., u_n$ and it can be calculated by the following formula:

$$H_n(u_0, u_1, u_2, ..., u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \qquad n = 0,1,2,3,....$$

Substituting equations (6) and (7) in equation (5), using HPM [18-19], we have

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = h(x) - p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u) \right]\right]\right). \tag{8}$$

Equating the coefficient of like power of p on both sides of equation (8), the following approximations are determined as follow

$$\begin{aligned} &p^0 : u_0(x,t) = h(x), \\ &p^n : u_n(x,t) = -L^{-1}\left(\frac{1}{s^{\alpha}} L[Ru_{n-1}(x,t) + H_{n-1}(u)] \right), \ n > 0, \ n \in N. \end{aligned} \tag{9}$$

Proceeding in this same manner, the rest of the components $u_n(x,t)$ can be completely obtained and the series solution is thus entirely found.

Finally, we approximate the solution $u(x,t)$ by truncated series

$$u(x,t) = \lim_{N \to \infty} \sum_{n=1}^{N} u_n(x,t) \tag{10}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [25].

6. Applications and Numerical Results

Consider the following generalized Hirota-Satsuma Coupled KdV equation [26],

$$\left. \begin{aligned} &\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u\frac{\partial^3 u}{\partial x^3} u_x + 3(vw)_x, \\ &\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = -\frac{\partial^3 v}{\partial x^3} + 3u\frac{\partial v}{\partial x}, \\ &\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = -\frac{\partial^3 w}{\partial x^3} + 3u\frac{\partial w}{\partial x}, \end{aligned} \right\}, \tag{11}$$

with initial conditions are:

$$\left.\begin{aligned} u(x,0) &= \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx), \\ v(x,0) &= -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx), \\ w(x,0) &= c_0 + c_1\tanh(kx) \end{aligned}\right\}, \tag{12}$$

Applying the Laplace Transform with initial conditions (12) in (11) we obtain

$$\left.\begin{aligned} u(x,s) &= \frac{1}{s}\left[\frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx)\right] + \frac{1}{s^\alpha}L\left[\frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x\right], \\ v(x,s) &= \frac{1}{s}\left[-\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx)\right] + \frac{1}{s^\alpha}L[-v_{xxx} + 3uv_x], \\ w(x,s) &= \frac{1}{s}[c_0 + c_1\tanh(kx)] + \frac{1}{s^\alpha}L[-w_{xxx} + 3uw_x], \end{aligned}\right\}$$

(13)

Where s is the parameter and t is the time.

Further, taking inverse Laplace Transform of equation (13), then we have

$$\left.\begin{aligned} u(x,t) &= \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx) + L^{-1}\left(\frac{1}{s^\alpha}L\left[\frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x\right]\right), \\ v(x,t) &= -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx) + L^{-1}\left(\frac{1}{s^\alpha}L[-v_{xxx} + 3uv_x]\right), \\ w(x,t) &= c_0 + c_1\tanh(kx) + L^{-1}\left(\frac{1}{s^\alpha}L[-w_{xxx} + 3uw_x]\right), \end{aligned}\right\}$$

(14)

According to the Homotopy perturbation method, we construct Homotopy of (14) yield

$$\left.\begin{aligned} \sum_{n=0}^{\infty} p^n u_n &= \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx) + p\left\{L^{-1}\left(\frac{1}{s^\alpha}L\left[\frac{1}{2}\sum_{n=0}^{\infty} p^n u_{xxx} + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right)\right\}, \\ \sum_{n=0}^{\infty} p^n v_n &= -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx) + pL^{-1}\left(\frac{1}{s^\alpha}L\left[\frac{1}{2}\sum_{n=0}^{\infty} p^n v_{xxx}\right]\right) \\ &\quad + pL^{-1}\left(\frac{1}{s^\alpha}L\left[\frac{1}{2}\sum_{n=0}^{\infty} p^n H_n(v)\right]\right), \\ \sum_{n=0}^{\infty} p^n w_n &= c_0 + c_1\tanh(kx) + pL^{-1}\left(\frac{1}{s^\alpha}L\left[-\sum_{n=0}^{\infty} p^n w_{xxx} + 3\sum_{n=0}^{\infty} p^n H_n(w)\right]\right) \end{aligned}\right\},$$

(15)

Equating the terms with identical powers of p of equation (15), we get

$$\left.\begin{aligned}
&p^0 : u_0 = \frac{1}{3}\left(\beta - 2k^2\right) + 2k^2 \tanh^2(kx),\\
&p^1 : u_1 = 4k^3\beta \sec h^2(kx)\tanh(kx)\frac{t^\alpha}{\Gamma(1+\alpha)},\\
&p^2 : u_2 = -4k^4\beta^2(-2+\cosh[2kx])\sec h^4(kx)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\
&p^3 : u_3 = -2k^5\beta^2\left[-104k^2 + \beta + 40k^2 coh(2kx) - \beta coh(4kx)\right]\sec h(kx)^6 \tan(4kx)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\\
&\qquad -2k^5\beta^2\left[\frac{4\left(13k^2+\beta+\left(-5k^2+\beta\right)\cosh(2kx)\right)\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)^2}\right]\sec h(kx)^6\tan(4kx)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}
\end{aligned}\right\}$$

(16)

so on for other components.

The m-th order approximate solution is given by

$$u(x,t) = \sum_{k=0}^{m} u_k(x,t)$$

Fig-1 shows the plot of $u(x,t)$ for $\alpha = 0.5$ and $\alpha = 0.75$.

The solution in a closed form when $\alpha = 1$, is given by

$$u(x,t) = \frac{1}{3}\left(\beta - 2k^2\right) + 2k^2 \tanh^2\left[k\left(x + \beta t\right)\right].$$

(17)

Equating the terms with identical powers of p for equation (15), again we have

$$\left.\begin{aligned}
&p^0 : v_0 = \frac{4k^2c_0\left(\beta+k^2\right)}{3c_1^2} + \frac{4k^2\left(\beta+k^2\right)}{3c_1}\tanh(kx),\\
&p^1 : v_1 = \frac{4k^3\beta\left(\beta+k^2\right)\sec h^2(kx)}{3c_1}\frac{t^\alpha}{\Gamma(1+\alpha)},\\
&p^2 : v_2 = \frac{8k^4\beta^2\left(\beta+k^2\right)\sec h^2(kx)}{3c_1}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\
&p^3 : v_3 = 2k^5\beta^2\left(\beta+k^2\right)\left[\frac{-48k^2-3\beta+\left(48k^2-2\beta\right)\cosh(2kx)+\beta\cosh(4kx)}{3c_1}\right]\\
&\qquad \sec h^6(kx)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - 96k^7\beta^2\left(\beta+k^2\right)\frac{\Gamma(1+2\alpha)\sinh^2(kx)}{3c_1\Gamma(1+\alpha)^2}\sec h^6(kx)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},
\end{aligned}\right\}$$

(18)

so on for other components.

The m-th order approximate solution is given by

$$v(x,t)=\sum_{k=0}^{m} v_k(x,t)$$

Fig-2 shows the plot of $v(x,t)$ for $\alpha = 0.5$ and $\alpha = 0.75$.

The solution in a closed form when $\alpha = 1$, is given by

$$v(x,t)=\frac{-4k^2 c_0\left(\beta+k^2\right)}{3c_1^2}+\frac{4k^2\tanh\left[k(x+\beta t)\right]\left(\beta+k^2\right)}{3c_1} \tag{19}$$

Similarly equating the terms with identical powers of p for equation (15), we have

$$\left.\begin{aligned}
&p^0 : w_0 = c_0 + c_1\tanh(kx),\\
&p^1 : w_1 = c_1 k\beta\,\mathrm{sech}^2(kx)\frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
&p^2 : w_2 = -2c_1k^2\beta^2\,\mathrm{sech}^2(kx)\tanh(kx)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\
&p^3 : w_3 = c_1k^3\beta^2\left[\frac{-48k^2-3\beta+\left(48k^2-2\beta\right)\cosh(2kx)+\beta\cosh(4kx)}{2}\right]-\\
&\qquad\qquad \frac{48k^2\Gamma(1+2\alpha)\sinh^2(kx)}{2\Gamma(1+2\alpha)^2}\mathrm{sech}^6(kx)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}
\end{aligned}\right\} \tag{20}$$

so on for other components.

The m-th order approximate solution is given by

$$w(x,t)=\sum_{k=0}^{m} w_k(x,t)$$

Fig-3 shows the plot of $w(x,t)$ for $\alpha = 0.5$ and $\alpha = 0.75$.

The solution in a closed form when $\alpha = 1$, is given by

$$w(x,t)=c_0+c_1\tanh\left[k(x+\beta t)\right] \tag{21}$$

7. Conclusions:

In this paper, the homotopy perturbation transforms method (HPTM) was used for finding solutions of a fractional order Hirota-Satsuma coupled KdV equation with initial conditions. In our work we obtained an approximate solution which converges to exact solution for different fractional order of dependent variable. We use the MATHEMATICA-6 to determine the series solution obtained from homotopy perturbation transforms method (HPTM).

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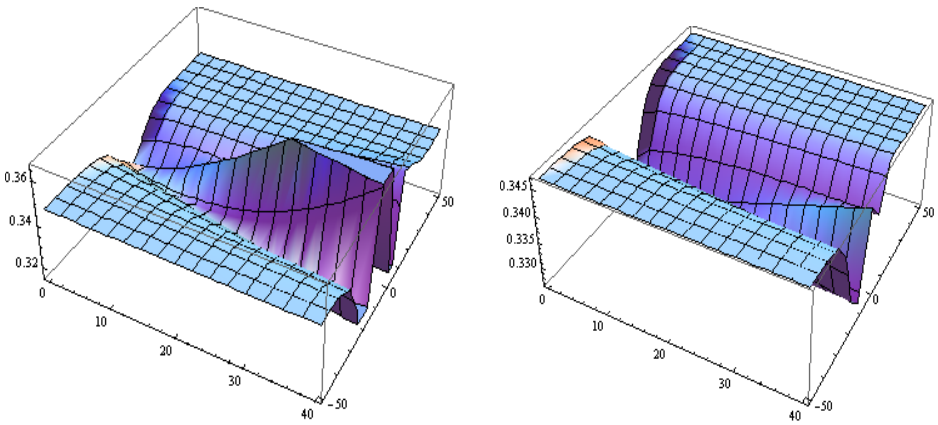
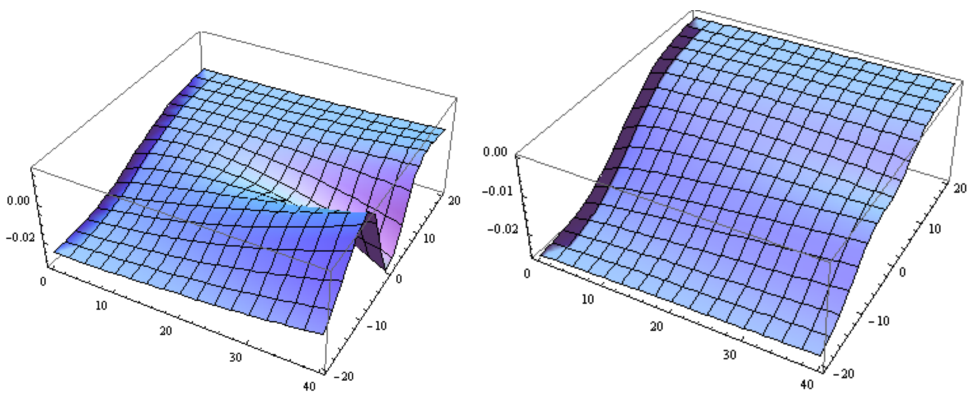


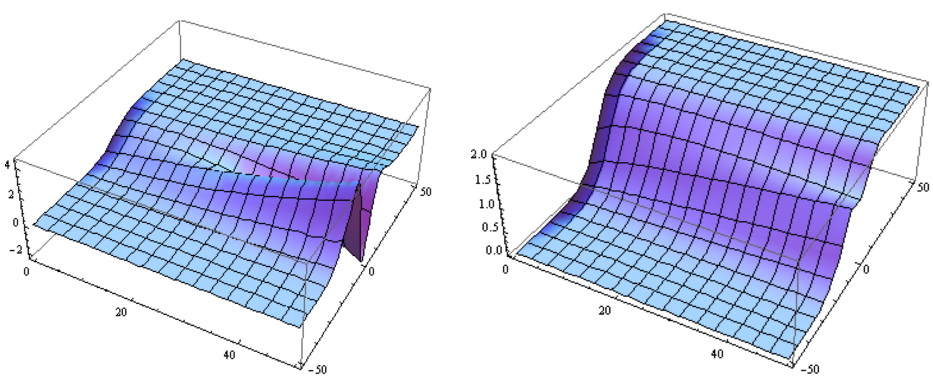
1(a) **1(b)**

Fig 1. Plot of $u(x,t)$ for $k=0.1, \beta=1$ and different values of α. Fig 1(a) for $\alpha=0.75$ Fig 1(b) for $\alpha=0.5$



2(a) 2(b)

Fig 2.Plot of $v(x,t)$ for $c_0=1,\ c_1=1\ k=0.1, \beta=1$, and different values of α. Fig 2(a) for $\alpha=0.75$ Fig 2(b) for $\alpha=0.5$



3(a) ` 3(b)

Fig 3. Plot of $w(x,t)$ for $k=0.1, c_0=1, c_1=1, \beta=0.1$ and different values of α. Fig 3(a) for $\alpha=0.75$ Fig 3(b) for $\alpha=0.5$