Rainbow connection number of Connected Graph

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Abstract: *A path of a connected graph G is rainbow if no two edges of it are colored the same. An edge-coloring graph G is rainbow connected if a rainbow path connects any two vertices. An edge coloring under which G is rainbow connected is called a rainbow coloring. The rainbow connection number of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. In this paper, we construct different type of connected graph and determine the exact value of $rc(G)$ for them.*

I. Introduction

Connectivity is a fundamental graph theoretic property. There are many ways to strengthen the connectivity property such as hamiltonicity, k −connectivity, requiring the existence of edge-disjoint spanning trees and so on. A natural way to strengthen the connectivity requirement was introduced by Chartrand et al. in [4]. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1, 2, \dots, n\}, n \in N$, is defined, where adjacent edges may be colored the same. A path is rainbow if no two edges of it are colored the same. An edge-coloring graph G is rainbow connected if a rainbow path connects any two vertices. Clearly, if a graph is rainbow edge-connected, then it is also connected. An edge coloring under which G is rainbow connected is called a rainbow coloring.

Any connected graph has trivial edge coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the rainbow connection number of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. Our main aim in this paper is to study the extremal graph-theoretic behavior of rainbow connection. Besides its theoretical interest, rainbow connectivity is also of interest in applied settings, such as securing sensitive information, transfer and networking, routing messages on cellular networks. In their original paper Chartrand et al. [4] determined $rc(G)$ in special cases where G is complete bipartite or multipartite graph. Rainbow connectivity for the computational point of view was first studied by Caro et al.[12] who conjectured that computing the rainbow connection number of a given graph is NP-hard. This conjecture was confirmed by Chakraborty et al. [10] who proved even deciding whether rainbow connection number of a graph equals 2 is NP-Complete. They further showed that the problem of deciding whether rainbow connection number of a graph is almost k is NP-hard, where k is an even integer.

II PRELIMINARIES

We shall use the standard terminology of graph theory as it is introduced in most of the text books on the theory of graph. Two vertices in G are said to be adjacent (or neighbours) if they are connected by an edge. The number of adjacent vertices of a vertex v_i, denoted by d_i is its degree. We denote the minimum degree of vertex by $\delta(G)$. An edge is a bridge if and only if it is not contained in any cycle. Diameter of a graph G denoted by $diam(G)$, is the longest distance between two vertices in a graph. A graph G is said to be minimally connected if the removal of an edge will disconnects the graph. A subgraph H spans a graph G and is a spanning subgraph of G, if it has the same vertex set as G.

III Related Work And Main Results

Chartrand et. al.[4] introduced the notion of the rainbow coloring in 2008. Chartrand et.al. obtained that $rc(G) = 1$ if and only if G is complete graph as well as that a cycle with $k > 3$ vertices has rainbow connection number $\lceil \frac{k}{2} \rceil$, a triangle has rainbow connection number 1.

Proposition 1[4]. For each integer $n \geq 3$, we have

$$rc(Wn) = \begin{cases} 1 & if\ n = 3, \\ 2 & if\ 4 \leq n \leq 7, \\ 3 & if\ n > 7, \end{cases}$$

Now we construct graph G_e from a wheel graph W_n with a vertex added between each pair of neighboring vertices of the outer cycle such that G_e has $2n + 1$ vertices and $3n$ edges.

Theorem 2. Let G be the connected graph of order $n > 3$. If the ordered degree sequences of connected graph G and graph G_e are same then $rc(G) = 4$.

Proof. By definition of G_e, we have $diam(G_e) = 4$. Since the ordered degree sequences of graph G and G_e are same then, $rc(G) \geq diam(G_e) = 4$. Consider a cycle C_n of length $n > 2$ is a connected graph in which the degree of each vertex is 2 and $V(C_{2n}) = \{x_1, x_2, \dots, x_{2n}\}$ is a set of vertices in cycle C_{2n} such that $V(G_e) = V(C_{2n}) \cup \{x\}, x \in V(G)$. Consider the mapping $f: E(G_e) \to \{1, 2, 3, 4\}$ defined as

$$f(e) = \begin{cases} 1, & if\ e = x\, x_{4i-3}\ for\ i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil, \\ 2, & if\ e = x\, x_{4i-1}\ for\ i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil, \\ 3, & if\ e = x_{4i-2}x_{4i-3}\ and\ e = x_{4i+1}x_{4i}\ for\ i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil, \\ 4, & if\ e = x_{4i-1}\, x_{4i-2}\ and\ e = x_{4i-1}x_{4i}\ for\ i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil \end{cases}$$

Clearly, f is a $4-$coloring of all the edges on G_e Thus, $rc(G_e) \leq 4$ and the theorem follows. ■

Again, we prove the analogous result.

Observation 3. Let G be the connected graph of order $n > 3$. If the ordered degree sequences of connected graph G and G' are same then $rc(G) = 4$, where G' is shown in figure 1:

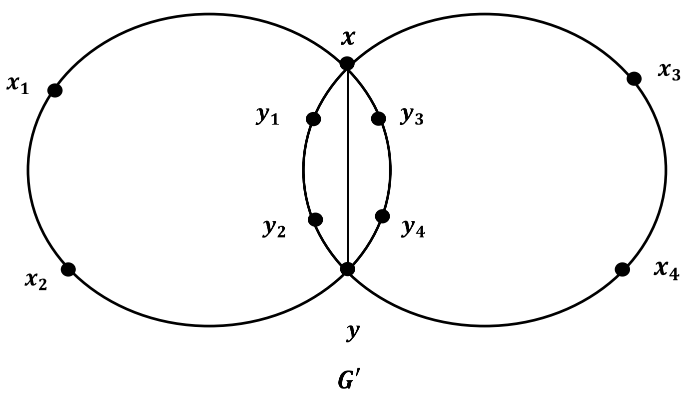

Figure 1

Proof. Since the ordered degree sequences of connected graph G and G' are same this implies $diam(G') = 3$. Let us assume $P := x_1, x, y, y_2$ be a rainbow path. Consider the mapping $f: E(G') \to \{1, 2, 3, 4\}$ defined as

$$f(e) = \begin{cases} 1, & if\ e = xy, \\ 2, & if\ e = xx_i\ for\ i = 1,2,3,4, \\ 3, & if\ e = yy_i\ for\ i = 1,2,3,4, \\ 4, & if\ e = x_iy_i\ for\ i = 1,2,3,4 \end{cases}$$

Clearly, f is a $4-$coloring of all the edges on G', Thus $rc(G') \leq 4$. Hence, the theorem is proved. ■

Theorem 4. If G is a connected graph of size l, then G is minimally connected if and only if $rc(G) = l$.

Proof. Let us assume G is not minimally connected then there exist a cycle $C_l: x_1, x_2, \dots, x_l, x_1$, where $l > 2$ in G such that $(l-1)$ coloring of the edges of G that assigns 1 to the edges x_1x_2 and x_2x_3 and assigns $l-2$ colors from $\{2, 3, \dots, l-1\}$ to the remaining edges of G is a rainbow coloring. Thus we have $rc(G) \leq l-1$, a contradiction.

Now suppose on contrary that $rc(G) \leq l-1$. Let m be the minimum rainbow coloring of G. Then there exist edges e_1 and e_2 such that $m(e_1) = m(e_2)$. Assume without loss of generality, that $e_1 = xy$ and $e_2 = uv$ and G contains a path $x, y, \dots, u, v$ then there exist no rainbow $x, y, \dots, u, v$ path in G, a contradiction. Hence we have, $rc(G) = l$. ■

Theorem 5. If G is a connected graph of size l, then all the edges of G are bridges if and if only if $rc(G) = l$.

Proof. Proof of the theorem is follows from theorem 4. ■

Caro et. al. investigated the extremal graph- theoretic behavior of rainbow connection number.

Theorem 6[12]. If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{5}{6}n$.

In the proof of theorem 6, Caro et. al. first gave an upper bound for the rainbow connection number of $2-$connected graphs then from it, they next derived an upper bound for the rainbow connection number of connected bridgeless graphs. They also showed the following upper bounds in term of minimum degree.

Theorem7[12]. If G is a connected graph with degree n vertices and minimum degree $\delta(G)$ then

$rc(G) \leq min\{n\frac{\ln(\delta(G))}{\delta(G)}(1 + o_{\delta(G)}(1)), n\frac{4\ln\delta(G)+3}{\delta(G)}\}$

Krivelevich and Yuster use the concept of connected two-step dominating set. They derived a rainbow coloring for G by giving a rainbow coloring to each subgraphs according to its connected two – step dominating set and gave the following theorem.

Theorem 8[6]. A connected graph G with n vertices has $rc(G) < 20n/(\delta(G))$.

2-constructible graph

A graph G is said to be $2-$constructible when it formed in one of the following three ways:

- The complete graph K_2 is $2-$constructible.
- Let G and H be any two $2-$constructible graphs. Then the graph formed by applying the Hajós construction to G and H is $2-$constructible.
- Let G be any $2-$constructible graph, and let x and y be any two non-adjacent vertices in G. Then the graph formed by combining x and y into a single vertex is also $2-$constructible. Equivalently, this graph may be formed by adding edge xy to the graph and then contracting it.

Theorem 9. If G is a $2-$constructible graph then $rc(G) \leq 2|V(G)|/3$.

Proof. Let H be a maximal connected subgraph of G satisfying the condition $rc(H) \leq 2|V(H)|/3 - 2/3$. We first claim that H exists. Indeed, if G has a triangle then taking H to be a triangle we have, $rc(H) = 1 \leq 2 - 2/3$. If G has any cycle of length $l \geq 4$ and $l \neq 5$, then already taking H to be such a cycle we obtain $rc(H) \leq 2l/3 - 2/3$. Otherwise, if each cycle of G is a C_5 then taking H to be a C_5 attached to one additional edge we get $rc(H) = 3 \leq 4 - 2/3, |V(H)| = 6$.

We next claim that $|V(H)| \geq |V(G)| - 2$. Let us assume first that there are three distinct vertices outside of H, say x_1, x_2, x_3, each having two adjacent vertices in H. Construct a new subgraph H_1 by adding k vertices of H with x_1, x_2 and x_3. Suppose $e_{i,}$ e_i' are two edges connecting x_i with H. We use only two new colors to color the edges, $e_1, e_2,$ e_3 all get the same color and e_1' , e_2', e_3' all get the same color. Thus we have,

$$rc(H_1) \leq rc(H) + 2 \leq 2|V(H)|/3 - 2/3 + 2 = 2(|V(H)| + 3)/3 - 2/3$$

Contradicting the maximality of H. It follows that if there are three vertices outside of H then at least one of these vertices, say x, has the property that a shortest path passing through x has length at least 3(as the graph is $2-$constructible there must be such a path). Let, therefore $y, z \in H$ and $x_1, \dots, x_m \notin H$ such that $y, x_1, \dots, x_m, z$ be a path from y to z. Construct a graph H_2 with $|V(H_2)| = |V(H)| + m$ by adding $x_1, \dots, x_m$ vertices to H. If m is even, we can color the $m+1$ edges of the path with $m/2$ colors as follows. The middle edge $(x_{m/2},\ x_{\frac{m}{2}+1})$ receives any color that already appears in H. The first $m/2$ edges of the path all receive distinct new colors and in the last $m/2$ edges of the path this coloring is repeated in the same order. Again, it is easy to verify that H_1 is rainbow connected.

If m is odd we can color the $m+1$ edges of the path with $(m+1)/2$ new colors. In the first half of the path the colors are all distinct, and the same ordering of colors is repeated in the second half of the path. It is easy to verify that H_1 is rainbow connected. We now have

$$rc(H_1) \leq rc(H) + \lceil m/2 \rceil \leq 2|V(H)|/3 - 2/3 + \lceil m/2 \rceil = 2(|V(H)| + m)/3 - 2/3$$

Contradicting the maximality of H. Clearly we have $rc(G) \leq 2(|V(G)| - 2)/3 - 2/3 + 2 = \frac{2|V(G)|}{3}$ as desired. If G is a $5-$cycle then $rc(G) = 3$ so the theorem clearly holds in this case. ∎

Again, we prove an upper bound for the $2-$ constructible graph and conclude the section with the following theorem.

Theorem 10. If G is a $2-$constructible graph then $rc(G) \leq |V(G)|/2 + O(\sqrt{|V(G)|})$.

Proof. A desirably edge coloring of G will be constructed consecutively, starting with a cycle on some number m of vertices and making it rainbow connected with $\lceil m/2 \rceil \leq m/2 + 1/2$ colors. Consider H_1 be the rainbow connected subgraph of G with $|V(H)|/2 + k$ colors and l denote the length of longest path that can currently be added, and also let $P = y, x_1, \dots, x_{l-1}, z$ be such a path, having its end vertices y, z in $H_1$and x_i's are outside H_1. Observe that no path of length l or more and being internally disjoint from $H_1 \cup P$ can join any x_i to any vertex of $H_1 \cup P$, otherwise l would not be maximum. Consequently, inserted paths of the same length are internally vertex-disjoint, and once we finish with l, we can never return to length l or more.

If l is odd, the value of k remains the same after insertion. Suppose that l is even. If just one or two paths of length l can be added before we continue the procedure with length $l-1$, then applying the coloring pattern as above, the value of k increases with ½ or 1, respectively. If three or more paths of length l are added, we multicolor all those paths with the same l new colors. In this way a rainbow-connected subgraph is obtained. For three or more paths we insert at least $3l - 3$ internal vertices, but use just $l < (31 - 3)/2$ colors if $l \geq 4$, what makes k decreases. If $l = 2$, k does not increase unless three paths are inserted, in which case k increases with ½.

The worst case for counting this upper bound on $rc(G) - |V(G)|/2$ is when each even path length 2, $4, \dots, n$ occurs precisely two times with no odd length. Even then, $rc(G) - |V(G)|/2 \leq n/2$ holds. Since $\sum_{j=1}^{n/2}(2j-1) < |V(G)|/2$ is valid for this particular sequence, we get n= $O(\sqrt{|V(G)|})$ as desired. ∎

Acknowledgement

The two authors would like to thank Dr. K.N. Rajeswari for the useful discussions on the topic.

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