Finite series relations involving the multivariable Aleph-function I

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Abstract
The aim of this document is to establish finite series relations for the multivariable Aleph-function. These relations are quite general in nature, from which a large number of new results can be obtained simply by specializing the parameters.

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1. Introduction and preliminaries.

The object of this document is to evaluate three finite double summations involving the multivariable aleph-function. These function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

where $j=1$ to r and $k=1$ to r. Suppose , as usual , that the parameters

$a_j,j=1,\cdots,p;b_j,j=1,\cdots,q;$

$c_j^{(k)},j=1,\cdots,n_k;c_{ji^{(k)}}^{(k)},j=n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j=1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,q_{i^{(k)}}$ with $k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's,\beta's,\gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i=1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)$ with $j=1$ to m_k are separated from those of $\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j=1$ to n and $\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)$ with $j=1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0, \text{ with } k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1}\ldots|z_r|^{\alpha_r}\,), max(\,|z_1|\ldots|z_r|\,)\rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1}\ldots|z_r|^{\beta_r}\,), min(\,|z_1|\ldots|z_r|\,)\rightarrow \infty$$

where, with $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})], j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U=p_i,q_i,\tau_i;R\ ;\ V=m_1,n_1;\cdots;m_r,n_r \tag{1.6}$$

$$\mathrm{W}=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.7}$$

$$A=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \quad (1.10)$$

$$D=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \quad (1.11)$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r)=\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\text{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \text{z}_r\end{array}\middle|\begin{array}{c}\text{A : C}\\ \cdot\ \cdot\ \cdot\\ \text{B : D}\end{array}\right) \quad (1.12)$$

2. Series relations

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b-c-n+1)_s(b)_t}{(2-e-m-n)_s(c)_t s!t!}\aleph_{p_i+1,q_i+2,\tau_i;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c}\text{z}_1\\ \cdot\\ \cdot\\ \text{z}_r\end{array}\middle|\begin{array}{c}(\text{1-c-s-t };\rho_1,\cdots,\rho_r),\\ \cdot\ \cdot\ \cdot\\ (\text{1-c-s };\rho_1,\cdots,\rho_r),\end{array}\right.$$

$$\left.\begin{array}{c}\text{A : C },\\ \cdot\ \cdot\ \cdot\\ (\text{1-c-t };\rho_1,\cdots,\rho_r),B:D\end{array}\right)=\frac{(b)_m(e-b)_n}{(e+n-1)_m(e)_n}\aleph_{p_i+1,q_i+2,\tau_i;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c}\text{z}_1\\ \cdot\\ \cdot\\ \text{z}_r\end{array}\middle|\begin{array}{c}(\text{1-c-m-n };\rho_1,\cdots,\rho_r),\\ \cdot\ \cdot\ \cdot\\ (\text{1-c-m };\rho_1,\cdots,\rho_r),\end{array}\right.$$

$$\left.\begin{array}{c}\text{A : C },\\ \cdot\ \cdot\ \cdot\\ (\text{1-c-n };\rho_1,\cdots,\rho_r),B:D\end{array}\right) \quad (2.1)$$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_s}{s!t!}\aleph_{p_i+3,q_i+3,\tau_i;R:W}^{m,\mathfrak{n}+2:V}\left(\begin{array}{c}\text{z}_1\\ \cdot\\ \cdot\\ \text{z}_r\end{array}\middle|\begin{array}{c}(\text{1-c-s-t };\rho_1,\cdots,\rho_r),\ (\text{d+s};\rho_1,\cdots,\rho_r),\\ \cdot\ \cdot\ \cdot\qquad\qquad \cdot\ \cdot\ \cdot\\ (\text{1-c-s };\rho_1,\cdots,\rho_r),\ (\text{1-c-t };\rho_1,\cdots,\rho_r),\end{array}\right.$$

$$\left.\begin{array}{c}(\text{d+m-a-s};\rho_1,\cdots,\rho_r),A:C\\ \cdot\ \cdot\ \cdot\\ (\text{d+m+n-t-1 };\rho_1,\cdots,\rho_r),B:D\end{array}\right)$$

$$=(-1)^{m+n}(a)_n\aleph_{p_i+3,q_i+3,\tau_i;R:W}^{m,\mathfrak{n}+2:V}\left(\begin{array}{c}\text{z}_1\\ \cdot\\ \cdot\\ \text{z}_r\end{array}\middle|\begin{array}{c}(\text{1-c-m-n };\rho_1,\cdots,\rho_r),\ (\text{d-a};\ \rho_1,\cdots,\rho_r),\\ \cdot\ \cdot\ \cdot\qquad\qquad \cdot\ \cdot\ \cdot\\ (\text{1-c-m };\rho_1,\cdots,\rho_r),\ (\text{1-c-n };\rho_1,\cdots,\rho_r),\end{array}\right.$$

$$\left.\begin{array}{c}(\text{d+m};\ \rho_1,\cdots,\rho_r),A:C\\ \cdot\ \cdot\ \cdot\\ (\text{d+m-1 };\rho_1,\cdots,\rho_r),B:D\end{array}\right) \quad (2.2)$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_t}{s!t!}\aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & (\text{1-c-s-t };\rho_1,\cdots,\rho_r),\ \ (\text{e+n-b-s };\rho_1,\cdots,\rho_r), \\ \cdot & \cdots \qquad\qquad \cdots \\ \cdot & (\text{1-c-s };\rho_1,\cdots,\rho_r),\ (\text{e+m+n-s-1 };\rho_1,\cdots,\rho_r), \\ \mathrm{z}_r & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{e+t };\rho_1,\cdots,\rho_r), A:C \\ \cdots \\ (\text{1-c-t };\rho_1,\cdots,\rho_r), B:D \end{array}\right)$$

$$= (-1)^{m+n}(b)_m\aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & (\text{1-c-m-n };\rho_1,\cdots,\rho_r),\ (\text{e-b };\rho_1,\cdots,\rho_r), \\ \cdot & \cdots \qquad\qquad \cdots \\ \cdot & (\text{1-c-m };\rho_1,\cdots,\rho_r),\ (\text{1-c-n };\rho_1,\cdots,\rho_r), \\ \mathrm{z}_r & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{e+n};\rho_1,\cdots,\rho_r), A:C, \\ \cdots \\ (\text{e+n-1 };\rho_1,\cdots,\rho_r), B:D \end{array}\right) \tag{2.3}$$

Conditions corresponding appropriately to the conditions (1.4) and (1.5) are assumed to be satisfied by all the multivariable Aleph-function occuring in the above formulas

Proof of (2.1)

Expressing the multivariable Aleph-function in the sum on the left of (2.1) in terms of its Mellin-Barnes type contour integral (1.1), interchanging the order of integration ans summation, which is valid, as the series involved is finite and evaluating the inner double series with the help of the following known result of Carlitz [1,p.139] :

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(a)_s(b)_t(c)_{s+t}}{(d)_s(e)_t(c)_s(c)_t t!s!} = \frac{(b)_m(a)_n(c)_{m+n}}{(b-a)_m(a-b)_n(c)_m(c)_n} \tag{2.4}$$

$(a-b$ is not an integer, $a=b-e-n+1$ and , $b=a-d-m+1)$, the right hand side of (2.1) is obtained, on interpreting the result.

The result (2.2) and (2.3) can be obtained by the similar method.

Remark

On taking $n=b=0$ in (2.3), the double serie reduces to the following simple serie relation for Aleph-function of several variables.

$$\sum_{s=0}^{m}\frac{(-m)_s}{s!}\aleph^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & (\text{e-s };\rho_1,\cdots,\rho_r), A:B \\ \cdot & \cdots \\ \cdot & (\text{e+m-s-1 };\rho_1,\cdots,\rho_r), C:D \\ \mathrm{z}_r & \end{array}\right)$$

$$= (-1)^m\aleph^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & (\text{e};\ \rho_1,\cdots,\rho_r), A:C, \\ \cdot & \cdots \\ \cdot & (\text{e-1 };\rho_1,\cdots,\rho_r), B:D \\ \mathrm{z}_r & \end{array}\right) \tag{2.5}$$

3. Multivariable I-function

If , $\tau_i = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = 1$the Aleph-function of several variables degenere to the I-function of several variables. The finite double sums have been derived in this section for multivariable I-functions defined by Sharma et al [2]. In these section, we note

$$B_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R \;, i^{(k)} = 1, \cdots, R^{(k)} \qquad (3.1)$$

Formula 1

$$\sum_{s=0}^{m} \sum_{t=0}^{n} \frac{(-m)_s(-n)_t(b-c-n+1)_s(b)_t}{(2-e-m-n)_s(c)_t s!t!} I_{p_i+1,q_i+2;R:W}^{m,\mathfrak{n}+1:V} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array} \middle| \begin{array}{c} (\text{1-c-s-t } ;\rho_1, \cdots, \rho_r), \\ \cdots \\ (\text{1-c-s } ;\rho_1, \cdots, \rho_r), \end{array} \right.$$

$$\left. \begin{array}{c} \text{A : C ,} \\ \cdots \\ (\text{1-c-t } ;\rho_1, \cdots, \rho_r), B : D \end{array} \right) = \frac{(b)_m(e-b)_n}{(e+n-1)_m(e)_n} I_{p_i+1,q_i+2;R:W}^{m,\mathfrak{n}+1:V} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array} \middle| \begin{array}{c} (\text{1-c-m-n } ;\rho_1, \cdots, \rho_r), \\ \cdots \\ (\text{1-c-m } ;\rho_1, \cdots, \rho_r), \end{array} \right.$$

$$\left. \begin{array}{c} \text{A : C ,} \\ \cdots \\ (\text{1-c-n } ;\rho_1, \cdots, \rho_r), B : D \end{array} \right) \qquad (3.2)$$

Formula 2

$$\sum_{s=0}^{m} \sum_{t=0}^{n} \frac{(-m)_s(-n)_t(b)_s}{s!t!} I_{p_i+3,q_i+3;R:W}^{m,\mathfrak{n}+2:V} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array} \middle| \begin{array}{c} (\text{1-c-s-t } ;\rho_1, \cdots, \rho_r), \; (\text{d+s};\rho_1, \cdots, \rho_r), \\ \cdots \qquad\qquad \cdots \\ (\text{1-c-s } ;\rho_1, \cdots, \rho_r), \; (\text{1-c-t } ;\rho_1, \cdots, \rho_r), \end{array} \right.$$

$$\left. \begin{array}{c} (\text{d+m-a-s};\rho_1, \cdots, \rho_r), A : C \\ \cdots \\ (\text{d+m+n-t-1 } ;\rho_1, \cdots, \rho_r), B : D \end{array} \right)$$

$$= (-1)^{m+n}(a)_n I_{p_i+3,q_i+3;R:W}^{m,\mathfrak{n}+2:V} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array} \middle| \begin{array}{c} (\text{1-c-m-n } ;\rho_1, \cdots, \rho_r), \; (\text{d-a}; \; \rho_1, \cdots, \rho_r), \\ \cdots \qquad\qquad \cdots \\ (\text{1-c-m } ;\rho_1, \cdots, \rho_r), \; (\text{1-c-n } ;\rho_1, \cdots, \rho_r), \end{array} \right.$$

$$\left. \begin{array}{c} (\text{d+m}; \; \rho_1, \cdots, \rho_r), A : C \\ \cdots \\ (\text{d+m-1 } ;\rho_1, \cdots, \rho_r), B : D \end{array} \right) \qquad (3.3)$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_t}{s!t!}I^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3;R:W}\left(\begin{array}{c|l} \mathrm{z}_1 & (\text{1-c-s-t };\rho_1,\cdots,\rho_r),\ \ (\text{e+n-b-s };\rho_1,\cdots,\rho_r), \\ \cdot & \qquad\qquad \cdots \qquad\qquad\qquad\quad \cdots \\ \cdot & \ (\text{1-c-s };\rho_1,\cdots,\rho_r),\ (\text{e+m+n-s-1 };\rho_1,\cdots,\rho_r), \\ \mathrm{z}_r & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{e+t };\rho_1,\cdots,\rho_r),A:C \\ \cdots \\ (\text{1-c-t };\rho_1,\cdots,\rho_r),B:D \end{array}\right)$$

$$=(-1)^{m+n}(b)_m\,I^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3;R:W}\left(\begin{array}{c|l} \mathrm{z}_1 & (\text{1-c-m-n };\rho_1,\cdots,\rho_r),\ (\text{e-b };\rho_1,\cdots,\rho_r), \\ \cdot & \qquad\qquad \cdots \qquad\qquad\qquad \cdots \\ \cdot & \ (\text{1-c-m };\rho_1,\cdots,\rho_r),\ (\text{1-c-n };\rho_1,\cdots,\rho_r), \\ \mathrm{z}_r & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{e+n};\rho_1,\cdots,\rho_r),A:C \\ \cdots \\ (\text{e+n-1 };\rho_1,\cdots,\rho_r),B:D \end{array}\right) \tag{3.4}$$

4. Aleph-function of two variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [4], and we have the following relations.

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b-c-n+1)_s(b)_t}{(2-e-m-n)_s(c)_ts!t!}\aleph^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2,\tau_i;R:W}\left(\begin{array}{c|l} \mathrm{z}_1 & (\text{1-c-s-t };\rho_1\rho_2),\qquad\quad \text{A : C },\\ \cdot & \qquad\quad \cdots \qquad\qquad\qquad\ \cdots \\ \cdot & (\text{1-c-s };\rho_1,\rho_2),(\text{1-c-t };\rho_1,\rho_2),B:D \\ \mathrm{z}_2 & \end{array}\right)$$

$$=\frac{(b)_m(e-b)_n}{(e+n-1)_m(e)_n}\aleph^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2,\tau_i;R:W}\left(\begin{array}{c|l} \mathrm{z}_1 & (\text{1-c-m-n };\rho_1,\rho_2),\qquad\quad \text{A : C },\\ \cdot & \qquad\quad \cdots \qquad\qquad\qquad\ \cdots \\ \cdot & \ (\text{1-c-m };\rho_1,\rho_2),\ (\text{1-c-n };\rho_1,\rho_2),B:D \\ \mathrm{z}_2 & \end{array}\right) \tag{4.1}$$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_s}{s!t!}\aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|l} \mathrm{z}_1 & (\text{1-c-s-t };\rho_1,\rho_2),\ (\text{d+s};\rho_1,\rho_2),\\ \cdot & \qquad\quad \cdots \qquad\qquad\qquad \cdots \\ \cdot & \ (\text{1-c-s };\rho_1,\rho_2),\ (\text{1-c-t };\rho_1,\rho_2), \\ \mathrm{z}_2 & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{d+m-a-s};\rho_1,\rho_2),A:C \\ \cdots \\ (\text{d+m+n-t-1 };\rho_1,\rho_2),B:D \end{array}\right)=(-1)^{m+n}(a)_n\aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|l} \mathrm{z}_1 & (\text{1-c-m-n };\rho_1,\rho_2),\\ \cdot & \qquad\quad \cdots \\ \cdot & \ (\text{1-c-m };\rho_1,\rho_2), \\ \mathrm{z}_2 & \end{array}\right.$$

$$\left.\begin{array}{cc} (\text{d-a; }\rho_1,\rho_2), & (\text{d+m; }\rho_1,\rho_2),A:C \\ \cdots & \cdots \\ (\text{1-c-n };\rho_1,\rho_2),, & (\text{d+m-1 };\rho_1,\rho_2),B:D \end{array}\right) \tag{4.2}$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_t}{s!t!}\aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{array}\right|\begin{array}{cc} (\text{1-c-s-t};\rho_1,\rho_2), & (\text{e+n-b-s};\rho_1,\rho_2), \\ \cdots & \cdots \\ (\text{1-c-s};\rho_1,\rho_2), & (\text{e+m+n-s-1};\rho_1,\rho_2), \end{array}$$

$$\left.\begin{array}{c} (\text{e+t};\rho_1,\rho_2), A : C \\ \cdots \\ (\text{1-c-t};\rho_1,\rho_2), B : D \end{array}\right) = (-1)^{m+n}(b)_m\aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{array}\right|\begin{array}{cc} (\text{1-c-m-n};\rho_1,\rho_2), & (\text{e-b};\rho_1,\rho_2), \\ \cdots & \cdots \\ (\text{1-c-m};\rho_1,\rho_2), & (\text{1-c-n};\rho_1,\rho_2), \end{array}$$

$$\left.\begin{array}{c} (\text{e+n};\rho_1,\rho_2), A : C, \\ \cdots \\ (\text{e+n-1};\rho_1,\rho_2), B : D \end{array}\right) \tag{4.3}$$

with the same conditions $(r = 2)$

5. I-function of two variables

If $\tau_i = \tau_i' = \tau_i'' = 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [3] and we obtain

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b-c-n+1)_s(b)_t}{(2-e-m-n)_s(c)_t s!t!}I^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2;R:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{array}\right|\left.\begin{array}{cc} (\text{1-c-s-t};\rho_1\rho_2), & \mathrm{A} : \mathrm{C}\ , \\ \cdots & \cdots \\ (\text{1-c-s};\rho_1,\rho_2), (\text{1-c-t};\rho_1,\rho_2), & B : D \end{array}\right)$$

$$= \frac{(b)_m(e-b)_n}{(e+n-1)_m(e)_n}I^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2;R:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{array}\right|\left.\begin{array}{cc} (\text{1-c-m-n};\rho_1,\rho_2), & \mathrm{A} : \mathrm{C}\ , \\ \cdots & \cdots \\ (\text{1-c-m};\rho_1,\rho_2), & (\text{1-c-n};\rho_1,\rho_2), B : D \end{array}\right) \tag{5.1}$$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_s}{s!t!}I^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3;R:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{array}\right|\begin{array}{cc} (\text{1-c-s-t};\rho_1,\rho_2), & (\text{d+s};\rho_1,\rho_2), \\ \cdots & \cdots \\ (\text{1-c-s};\rho_1,\rho_2), & (\text{1-c-t};\rho_1,\rho_2), \end{array}$$

$$\left.\begin{array}{c} (\text{d+m-a-s};\rho_1,\rho_2), A : C \\ \cdots \\ (\text{d+m+n-t-1};\rho_1,\rho_2), B : D \end{array}\right) = (-1)^{m+n}(a)_n I^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+3;R:W}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{array}\right|\begin{array}{c} (\text{1-c-m-n};\rho_1,\rho_2), \\ \cdots \\ (\text{1-c-m};\rho_1,\rho_2), \end{array}$$

$$\left.\begin{array}{cc} (\text{d-a};\ \rho_1,\rho_2), & (\text{d+m};\ \rho_1,\rho_2), A : C \\ \cdots & \cdots \\ (\text{1-c-n};\rho_1,\rho_2),, & (\text{d+m-1};\rho_1,\rho_2), B : D \end{array}\right) \tag{5.2}$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_t}{s!t!}I^{m,\mathrm{n}+2:V}_{p_i+3,q_i+3;R:W}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \mathrm{z}_2\end{array}\middle|\begin{array}{c}(\text{1-c-s-t };\rho_1,\rho_2),\ (\text{e+n-b-s };\rho_1,\rho_2),\ (\text{e+t };\rho_1,\rho_2),A:C\\ \cdots\qquad\cdots\qquad\cdots\\ (\text{1-c-s };\rho_1,\rho_2),\ (\text{e+m+n-s-1 };\rho_1,\rho_2),\ (\text{1-c-t };\rho_1,\rho_2),B:D\end{array}\right)=(-1)^{m+n}(b)_m I^{m,\mathrm{n}+2:V}_{p_i+3,q_i+3;R:W}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \mathrm{z}_2\end{array}\middle|\begin{array}{c}(\text{1-c-m-n };\rho_1,\rho_2),\ (\text{e-b };\rho_1,\rho_2),\ (\text{e+n};\rho_1,\rho_2),A:C,\\ \cdots\qquad\cdots\qquad\cdots\\ (\text{1-c-m };\rho_1,\rho_2),\ (\text{1-c-n };\rho_1,\rho_2),\ (\text{e+n-1 };\rho_1,\rho_2),B:D\end{array}\right)\tag{5.3}$$

6. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function ,multivariable Fox's H-function, Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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