Finite series relations involving the multivariable Aleph-function III

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Abstract
The aim of this document is to establish finite series relations for the multivariable Aleph-function. These relations are quite general in nature, from which a large number of new results can be obtained simply by specializing the parameters.

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1. Introduction and preliminaries.

The object of this document is to evaluate three finite double summations involving the multivariable aleph-function. These function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [3] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $$\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$$

$$\begin{array}{l}[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:\\ \cdots\cdots\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

where $j=1$ to r and $k=1$ to r. Suppose , as usual , that the parameters

$a_j,j=1,\cdots,p;b_j,j=1,\cdots,q;$

$c_j^{(k)},j=1,\cdots,n_k;c_{ji^{(k)}}^{(k)},j=n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)},j=1,\cdots,m_k;d_{ji^{(k)}}^{(k)},j=m_k+1,\cdots,q_{i^{(k)}}$ with $k=1\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's,\beta's,\gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i=1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)$ with $j=1$ to m_k are separated from those of $\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j=1$ to n and $\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)$ with $j=1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|argz_k|<\frac{1}{2}A_i^{(k)}\pi \text{ , where}$$

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\text{ with } k=1\cdots,r,i=1,\cdots,R\text{ },i^{(k)}=1,\cdots,R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1}\dots|z_r|^{\alpha_r}\,)\,,max(\,|z_1|\dots|z_r|\,)\rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1}\dots|z_r|^{\beta_r}\,)\,,min(\,|z_1|\dots|z_r|\,)\rightarrow\infty$$

where, with $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U=p_i,q_i,\tau_i;R\text{ };\text{ }V=m_1,n_1;\cdots;m_r,n_r \tag{1.6}$$

$$\text{W}=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.7}$$

$$A=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \quad (1.11)$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} \text{A : C} \\ \cdots \\ \text{B : D} \end{array}\right) \quad (1.12)$$

2. Series relations

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(c)_s(d)_t}{(c+d)_{s+t}s!t!}\aleph_{p_i+1,q_i+2,\tau_i;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} (\text{1-b-s-t };\rho_1,\cdots,\rho_r), \quad \text{A : C} \\ \cdots \quad \cdots \\ (\text{1-b-s };\rho_1,\cdots,\rho_r), (\text{1-b-t };\rho_1,\cdots,\rho_r), B:D \end{array}\right)$$

$$= \frac{(d)_m(c)_n}{(c+d)_{m+n}}\aleph_{p_i+1,q_i+2,\tau_i;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} (\text{1-b-m-n };\rho_1,\cdots,\rho_r), \quad \text{A : C} \\ \cdots \quad \cdots \\ (\text{1-b-m };\rho_1,\cdots,\rho_r), (\text{1-b-n };\rho_1,\cdots,\rho_r), B:D \end{array}\right) \quad (2.1)$$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(d)_t}{s!t!}\aleph_{p_i+2,q_i+3,\tau_i;R:W}^{m,\mathfrak{n}+2:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} (\text{1-b-s-t };\rho_1,\cdots,\rho_r), (\text{1-c-s };\rho_1,\cdots,\rho_r), \\ \cdots \quad \cdots \\ (\text{1-b-s };\rho_1,\cdots,\rho_r), (\text{1-b-t };\rho_1,\cdots,\rho_r), \end{array}\right.$$

$$\left.\begin{array}{c} \text{A : C} \\ \cdots \\ (\text{1-c-d-s-t };\rho_1,\cdots,\rho_r), B:D \end{array}\right) = (d)_m\aleph_{p_i+2,q_i+3,\tau_i;R:W}^{m,\mathfrak{n}+2:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} (\text{1-b-m-n };\rho_1,\cdots,\rho_r), \\ \cdots \\ (\text{1-b-m };\rho_1,\cdots,\rho_r), \end{array}\right.$$

$$\left.\begin{array}{c} (\text{1-c-n };\rho_1,\cdots,\rho_r), \quad \text{A : C} \\ \cdots \quad \cdots \\ (\text{1-b-n };\rho_1,\cdots,\rho_r), (\text{1-c-d-m-n };\rho_1,\cdots,\rho_r), B:D \end{array}\right) \quad (2.2)$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_{s+t}(d)_t}{(b-d-m+1)_s s!t!}\aleph_{p_i+2,q_i+1,\tau_i;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} (\text{1-c-s };\rho_1,\cdots,\rho_r), \\ \cdots \\ (\text{1-c-d-s-t };\rho_1,\cdots,\rho_r), \end{array}\right.$$

$$\left.\begin{array}{c} (\text{b-c-n-t-1};\rho_1,\cdots,\rho_r), A:C \\ \cdots \\ \text{B : D} \end{array}\right)$$

$$= \frac{(-)^n (d)_m}{(d-b)_m} \aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+2,\tau_i;R:W} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} (\text{1-c-n};\rho_1,\cdots,\rho_r),(1+b-c-d-m-n;\rho_1,\cdots,\rho_r), \\ \cdots \\ (\text{1-c-d-m-n};\rho_1,\cdots,\rho_r) \end{array} \right.$$

$$\left. \begin{array}{c} (\text{1+b-c};\rho_1,\cdots,\rho_r), A:C \\ \cdots \\ (\text{1+b-c-d};\rho_1,\cdots,\rho_r), B:D \end{array} \right) \tag{2.3}$$

The above conditions, are assumed to be satisfied by the Aleph-function of several variables occurring in the series relations (2.1) to (2.3).

Proof of (2.1) to (2.3)
Using the result of Carlitz [1,p.232, Eq.(11)]

$$\sum_{s=0}^{m}\sum_{t=0}^{n} \frac{(-m)_s(-n)_t(b)_{s+t}(c)_s(d)_t}{(c+d)_{s+t}(b)_s(b)_t s!t!} = \frac{(b)_{m+n}(d)_m(c)_n}{(c+d)_{m+n}(b)_m(b)_n} \tag{2.4}$$

in (2.1) and (2.2) and another result of Carlitz [2,p.416, eq. (9)]

$$\sum_{s=0}^{m}\sum_{t=0}^{n} \frac{(-m)_s(-n)_t(b)_{s+t}(c)_s(d)_t}{(c+d)_{s+t}(b-c-n+1)_s(b-d-m+1)_t s!t!} = \frac{(c+d-b)_{m+n}(d)_m(c)_n}{(c+d)_{m+n}(d-b)_m(c-b)_n} \tag{2.5}$$

In (2.3) and integrating term by term, we obtain the desired formulae.

3.Particular case

If we take $n=0$ in (2.3), the double finite serie reduces to the single serie for the multivariable Aleph-function.

$$\sum_{s=0}^{m} \frac{(-m)_s(b)_s}{(b-d-m+1)_s s!} \aleph^{m,\mathfrak{n}+1:V}_{p_i+2,q_i+1,\tau_i;R:W} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{cc} (\text{1-c-s};\rho_1,\cdots,\rho_r), & (\text{b-c+1};\rho_1,\cdots,\rho_r), A:C \\ \cdots & \cdots \\ (\text{1-c-d-s};\rho_1,\cdots,\rho_r), & \text{B : D} \end{array} \right)$$

$$= \frac{(d)_m}{(d-b)_m} \aleph^{m,\mathfrak{n}+1:V}_{p_i+2,q_i+2,\tau_i;R:W} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} (\text{1-c+b};\rho_1,\cdots,\rho_r),(1+b-c-d-m;\rho_1,\cdots,\rho_r), A:C \\ \cdots \\ (\text{1-c-m};\rho_1,\cdots,\rho_r),(1+b-c-d;\rho_1,\cdots,\rho_r), B:D \end{array} \right) \tag{3.1}$$

4. Multivariable I-function

If $\tau_i = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = 1$ the Aleph-function of several variables degenere to the I-function of several variables. The finite double sums have been derived in this section for multivariable I-functions defined by Sharma et al [3]. In these section, we note

$$B_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_{i(k)}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with}\ k=1\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)} \tag{4.1}$$

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(c)_s(d)_t}{(c+d)_{s+t}s!t!}I_{p_i+1,q_i+2;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{cc}(\text{1-b-s-t};\rho_1,\cdots,\rho_r), & A:C\\ \cdots & \cdots\\ (\text{1-b-s};\rho_1,\cdots,\rho_r),\ (\text{1-b-t};\rho_1,\cdots,\rho_r), & B:D\end{array}\right)$$

$$=\frac{(d)_m(c)_n}{(c+d)_{m+n}}I_{p_i+1,q_i+2;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{cc}(\text{1-b-m-n};\rho_1,\cdots,\rho_r), & A:C\\ \cdots & \cdots\\ (\text{1-b-m};\rho_1,\cdots,\rho_r),\ (\text{1-b-n};\rho_1,\cdots,\rho_r), & B:D\end{array}\right) \tag{4.1}$$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(d)_t}{s!t!}I_{p_i+2,q_i+3;R:W}^{m,\mathfrak{n}+2:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{ccc}(\text{1-b-s-t};\rho_1,\cdots,\rho_r),(\text{1-c-s};\rho_1,\cdots,\rho_r), & & A:C\\ \cdots & \cdots & \cdots\\ (\text{1-b-s};\rho_1,\cdots,\rho_r),\ (\text{1-b-t};\rho_1,\cdots,\rho_r), & & (\text{1-c-d-s-t};\rho_1,\cdots,\rho_r),B:D\end{array}\right)$$

$$=(d)_m I_{p_i+2,q_i+3;R:W}^{m,\mathfrak{n}+2:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{ccc}(\text{1-b-m-n};\rho_1,\cdots,\rho_r), & (\text{1-c-n};\rho_1,\cdots,\rho_r), & A:C\\ \cdots & \cdots & \cdots\\ (\text{1-b-m};\rho_1,\cdots,\rho_r), & (\text{1-b-n};\rho_1,\cdots,\rho_r), & (\text{1-c-d-m-n};\rho_1,\cdots,\rho_r),B:D\end{array}\right) \tag{4.2}$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_{s+t}(d)_t}{(b-d-m+1)_s s!t!}I_{p_i+2,q_i+1;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{cc}(\text{1-c-s};\rho_1,\cdots,\rho_r), & (\text{b-c-n-t-1};\rho_1,\cdots,\rho_r),A:C\\ \cdots & \cdots\\ (\text{1-c-d-s-t};\rho_1,\cdots,\rho_r), & B:D\end{array}\right)$$

$$=\frac{(-)^n(d)_m}{(d-b)_m}I_{p_i+3,q_i+2;R:W}^{m,\mathfrak{n}+2:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(\text{1-c-n};\rho_1,\cdots,\rho_r),(1+b-c-d-m-n;\rho_1,\cdots,\rho_r),\\ \cdots\\ (\text{1-c-d-m-n};\rho_1,\cdots,\rho_r)\end{array}\right.$$

$$\left.\begin{matrix} (1\text{+b-c };\rho_1,\cdots,\rho_r),A:C \\ \cdots \\ (1\text{+b-c-d};\rho_1,\cdots,\rho_r),B:D \end{matrix}\right) \tag{4.3}$$

with the condition $|argz_k| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (4.1).

5. Aleph-function of two variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [5], and we have the following relations.

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(c)_s(d)_t}{(c+d)_{s+t}s!t!}\aleph^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle|\begin{matrix} (\text{1-b-s-t };\rho_1,\rho_2), & A:C \\ \cdots & \cdots \\ (\text{1-b-s };\rho_1,\rho_2),\ (\text{1-b-t };\rho_1,\rho_2), & B:D \end{matrix}\right)$$

$$=\frac{(d)_m(c)_n}{(c+d)_{m+n}}\aleph^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle|\begin{matrix} (\text{1-b-m-n };\rho_1,\rho_2), & A:C \\ \cdots & \cdots \\ (\text{1-b-m };\rho_1,\rho_2),\ (\text{1-b-n };\rho_1,\rho_2), & B:D \end{matrix}\right) \tag{5.1}$$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(d)_t}{s!t!}\aleph^{m,\mathfrak{n}+2:V}_{p_i+2,q_i+3,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle|\begin{matrix} (\text{1-b-s-t };\rho_1,\rho_2),(\text{1-c-s };\rho_1,\rho_2), \\ \cdots \quad \cdots \\ (\text{1-b-s };\rho_1,\rho_2),\ (\text{1-b-t };\rho_1,\rho_2), \end{matrix}\right.$$

$$\left.\begin{matrix} A:C \\ \cdots \\ (\text{1-c-d-s-t };\rho_1,\rho_2),B:D \end{matrix}\right)=(d)_m\aleph^{m,\mathfrak{n}+2:V}_{p_i+2,q_i+3,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle|\begin{matrix} (\text{1-b-m-n };\rho_1,\rho_2),(\text{1-c-n };\rho_1,\rho_2), \\ \cdots \quad \cdots \\ (\text{1-b-m };\rho_1,\rho_2),\ (\text{1-b-n };\rho_1,\rho_2), \end{matrix}\right.$$

$$\left.\begin{matrix} A:C \\ \cdots \\ (\text{1-c-d-m-n };\rho_1,\rho_2),B:D \end{matrix}\right) \tag{5.2}$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_{s+t}(d)_t}{(b-d-m+1)_s s!t!}\aleph^{m,\mathfrak{n}+1:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle|\begin{matrix} (\text{1-c-s };\rho_1,\rho_2), & (\text{b-c-n-t-1};\rho_1,\rho_2),A:C \\ \cdots & \cdots \\ (\text{1-c-d-s-t };\rho_1,\rho_2), & B:D \end{matrix}\right)$$

$$=\frac{(-)^n(d)_m}{(d-b)_m}\aleph^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+2,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle|\begin{matrix} (\text{1-c-n };\rho_1,\rho_2),(1+b-c-d-m-n;\rho_1,\rho_2), \\ \cdots \\ (\text{ 1-c-d-m-n};\rho_1,\rho_2) \end{matrix}\right.$$

$$\left.\begin{matrix} \text{(1+b-c };\rho_1,\rho_2), A:C \\ \cdots \\ \text{(1+b-c-d};\rho_1,\rho_2), B:D \end{matrix}\right) \tag{5.3}$$

6. I-function of two variables

If $\tau_i = \tau_i' = \tau_i'' = 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [4] and we obtain

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(c)_s(d)_t}{(c+d)_{s+t}s!t!} I^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle| \begin{matrix} \text{(1-b-s-t };\rho_1,\rho_2), \qquad A:C \\ \cdots \qquad \cdots \\ \text{(1-b-s };\rho_1,\rho_2),\ \text{(1-b-t };\rho_1,\rho_2), B:D \end{matrix}\right)$$

$$= \frac{(d)_m(c)_n}{(c+d)_{m+n}} I^{m,\mathfrak{n}+1:V}_{p_i+1,q_i+2;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle| \begin{matrix} \text{(1-b-m-n };\rho_1,\rho_2), \qquad A:C \\ \cdots \qquad \cdots \\ \text{(1-b-m };\rho_1,\rho_2),\ \text{(1-b-n };\rho_1,\rho_2), B:D \end{matrix}\right) \tag{6.1}$$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(d)_t}{s!t!} I^{m,\mathfrak{n}+2:V}_{p_i+2,q_i+3;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle| \begin{matrix} \text{(1-b-s-t };\rho_1,\rho_2),\text{(1-c-s };\rho_1,\rho_2), \\ \cdots \qquad \cdots \\ \text{(1-b-s };\rho_1,\rho_2),\ \text{(1-b-t };\rho_1,\rho_2), \end{matrix}\right.$$

$$\left.\begin{matrix} A:C \\ \cdots \\ \text{(1-c-d-s-t };\rho_1,\rho_2), B:D \end{matrix}\right) = (d)_m I^{m,\mathfrak{n}+2:V}_{p_i+2,q_i+3;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle| \begin{matrix} \text{(1-b-m-n };\rho_1,\rho_2),\text{(1-c-n };\rho_1,\rho_2), \\ \cdots \qquad \cdots \\ \text{(1-b-m };\rho_1,\rho_2),\ \text{(1-b-n };\rho_1,\rho_2), \end{matrix}\right.$$

$$\left.\begin{matrix} A:C \\ \cdots \\ \text{(1-c-d-m-n };\rho_1,\rho_2), B:D \end{matrix}\right) \tag{6.2}$$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-m)_s(-n)_t(b)_{s+t}(d)_t}{(b-d-m+1)_s s!t!} I^{m,\mathfrak{n}+1:V}_{p_i+2,q_i+1;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle| \begin{matrix} \text{(1-c-s };\rho_1,\rho_2),\quad \text{(b-c-n-t-1};\rho_1,\rho_2), A:C \\ \cdots \qquad \cdots \\ \text{(1-c-d-s-t };\rho_1,\rho_2), \qquad B:D \end{matrix}\right)$$

$$= \frac{(-)^n(d)_m}{(d-b)_m} I^{m,\mathfrak{n}+2:V}_{p_i+3,q_i+2;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix}\middle| \begin{matrix} \text{(1-c-n };\rho_1,\rho_2),(1+b-c-d-m-n;\rho_1,\rho_2), \\ \cdots \\ \text{(1-c-d-m-n};\rho_1,\rho_2) \end{matrix}\right.$$

$$\left.\begin{matrix}(1+\text{b-c};\rho_1,\rho_2),A:C\\ \cdots\\ (1+\text{b-c-d};\rho_1,\rho_2),B:D\end{matrix}\right) \tag{6.3}$$

7. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function ,multivariable Fox's H-function, Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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