Double series relations involving the generalized multivariable Aleph-function

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Abstract
The aim of this document is to establish four double series relations for the generalized multivariable Aleph-function. These relations are quite general in nature, from which a large number of new results can be obtained simply by specializing the parameters.

Keywords :double series , generalized Aleph-function of several variables , multivariable I-function, aleph-function of two variables,

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1. Introduction and preliminaries.

The object of this document is to evaluate four finite double summations involving the generalized multivariable aleph-function. These function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r) = \aleph^{m,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}] \quad ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$[(\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})_{1,m}] \quad ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$

$$\left. \begin{array}{l} [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\sum_{k=1}^{r}\beta_j^{(k)}s_j)\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

where $j=1$ to r and $k=1$ to r. Suppose , as usual , that the parameters

$a_j, j=1,\cdots,p; b_j, j=1,\cdots,q;$

$c_j^{(k)}, j=1,\cdots,n_k; c_{ji^{(k)}}^{(k)}, j=n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j=1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,q_{i^{(k)}}$ with $k=1\cdots,r, i=1,\cdots,R \;, i^{(k)}=1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=m+1}^{q_i} \beta_{ji}^{(k)}$$

$$- \sum_{j=1}^{m_k} \delta_j^{(k)} - \sum_{j=1}^{m} \beta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \sum_{j=1}^{m} \beta_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=m+1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1} \dots |z_r|^{\alpha_r}), max(|z_1| \dots |z_r|) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1} \dots |z_r|^{\beta_r}), min(|z_1| \dots |z_r|) \rightarrow \infty$$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \text{ ; } V = m_1, n_1; \cdots; m_r, n_r \tag{1.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{(b_j; \beta_j, \cdots, \beta_j)_{1,m}\} \text{ ; } \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r)=\aleph^{m,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c|c} z_1 & \mathrm{A : C} \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ z_r & \mathrm{B : D}\end{array}\right) \tag{1.12}$$

2. Series relations

We obtain the following double series relations of the generalized multivariable Aleph-function :

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(c)_{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}\aleph^{m+4,\mathfrak{n}:V}_{U_{43};R:}\left(\begin{array}{c|c} z_1 & (\mathrm{a+s+t}\ ;\rho_1,\cdots,\rho_r),(d+s;\rho'_1,\cdots,\rho'_r) \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & (\tfrac{1}{2}\mathrm{+s\ +t};\rho'_1,\cdots,\rho'_r), \\ z_r & \end{array}\right.$$

$$\left.\begin{array}{ccc} (1+\tfrac{1}{2}\mathrm{+s+t}\ ;\rho'_1,\cdots,\rho'_r), & (\mathrm{e+t};\rho'_1,\cdots,\rho'_r), & \mathrm{A : C} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ (\mathrm{1+a\text{-}b+s+t};\rho_1,\cdots,\rho_r), & (\mathrm{1+a\text{-}c+s+t};\rho_1,\cdots,\rho_r), & \mathrm{B : D}\end{array}\right)=\frac{\Gamma(1+a-d-e)}{\Gamma(1+a-b-d-e)}$$

$$\frac{\Gamma(1+a-b-c-d-e)}{\Gamma(1+a-c-d-e)}\aleph^{m+2,\mathfrak{n}:V}_{U_{21};R:}\left(\begin{array}{c|c} z_1 & (\mathrm{d}\ ;\rho'_1,\cdots,\rho'_r),(e;\rho'_1,\cdots,\rho'_r),A:C \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & (\mathrm{1+a\text{-}b\text{-}c}\ ;\rho_1,\cdots,\rho_r),B:D \\ z_r & \end{array}\right) \tag{2.1}$$

where $W_{44}=p_i+4;q_i+3;\tau_i;R$ and $W_{21}=p_i+2;q_i+1;\tau_i;R$ and $\rho'_i=\dfrac{\rho_i}{2},i=1,\cdots,r$

provided that the double series involved on the left hand side of (2.1) is absolutely convergent,

$Re(b+c+d+e-a)<1,\rho_k>0,k=1,\cdots,r$

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(d)_s(e)_t}{s!t!}\aleph^{m+4,\mathfrak{n}:V}_{U_{44};R:}\left(\begin{array}{c|c} z_1 & (\mathrm{a+s+t}\ ;\rho_1,\cdots,\rho_r),(b+s+t;\rho'_1,\cdots,\rho'_r), \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & (\mathrm{1+a\text{-}b+s\ +t};\rho'_1,\cdots,\rho'_r),(\tfrac{a}{2}+r+s;\rho'_1,\cdots,\rho'_r), \\ z_r & \end{array}\right.$$

$$\left.\begin{array}{ccc} (1+\tfrac{a}{2}\mathrm{+s+t}\ ;\rho'_1,\cdots,\rho'_r), & (\mathrm{c+s+t};\rho'_1,\cdots,\rho'_r), & \mathrm{A : C} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ (\mathrm{1+a\text{-}c+s+t};\rho'_1,\cdots,\rho'_r), & (\mathrm{1+a\text{-}d\text{-}e+s+t};\rho_1,\cdots,\rho_r), & \mathrm{B : D}\end{array}\right)=\frac{\Gamma(1+a-b-d-e)}{2\Gamma(1+a-c)}$$

$$\aleph^{m+2,\mathfrak{n}:V}_{U_{22};R:}\left(\begin{array}{c|c} z_1 & (\mathrm{b}\ ;\rho'_1,\cdots,\rho'_r),(c;\rho'_1,\cdots,\rho'_r),A:C \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & (\mathrm{1+a\text{-}b\text{-}d\text{-}e}\ ;\rho'_1,\cdots,\rho'_r),(1+a-c-d-e;\rho'_1,\cdots,\rho'_r),B:D \\ z_r & \end{array}\right) \tag{2.2}$$

where $U_{44}=p_i+4;q_i+4;\tau_i;R$ and $U_{22}=p_i+2;q_i+2;\tau_i;R$ and $\rho'_i=\dfrac{\rho_i}{2},i=1,\cdots,r$

provided that the double series involved on the left hand side of (2.2) is absolutely convergent,

$Re(1+a-b-c-d-e)>0, \rho_k>0, k=1,\cdots,r$

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(d)_s(e)_s}{(1+a-b)_{s+t}s!t!}\aleph_{U_{32};R:}^{m+3,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(\text{d+s+t};\rho_1,\cdots,\rho_r),(1+\frac{a}{2}+s+t;\rho'_1,\cdots,\rho'_r)\\ \cdots\\ (\frac{a}{2}\text{+s+t};\rho'_1,\cdots,\rho'_r),\end{array}\right.$$

$$\left.\begin{array}{c}(\text{b+s+t};\rho'_1,\cdots,\rho'_r),\quad \text{A : C}\\ \cdots\quad\cdots\\ (\text{1+a-d-e+s+t};\rho_1,\cdots,\rho_r),\text{B : D}\end{array}\right)=\frac{\Gamma(1+a-b)}{\Gamma(1+a-b-d-e)}\aleph_{U_{22};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(\text{b};\rho_1,\cdots,\rho_r),A:C\\ \cdots\\ \text{B : D}\end{array}\right)\quad(2.3)$$

where $U_{32}=p_i+3;q_i+2;\tau_i;R$ and $\rho'_i=\dfrac{\rho_i}{2}, i=1,\cdots,r$

provided that the double series involved on the left hand side of (2.3) is absolutely convergent$\rho_k>0, k=1,\cdots,r$

Formula 4

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}\aleph_{U_{42};R:}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(\text{a+s+t};\rho_1,\cdots,\rho_r),(d+s;\rho'_1,\cdots,\rho'_r)\\ \cdots\\ (\frac{a}{2}\text{+s +t};\rho'_1,\cdots,\rho'_r),\end{array}\right.$$

$$\left.\begin{array}{c}(1+\frac{a}{2}\text{+s+t};\rho'_1,\cdots,\rho'_r),(\text{e+s};\rho'_1,\cdots,\rho'_r),\text{A : C}\\ \cdots\quad\cdots\quad\cdots\\ (\text{1+a-b+s+t};\rho_1,\cdots,\rho_r),\quad \cdots,\quad \text{B : D}\end{array}\right)=\frac{\Gamma(1+a-d-e)}{2\Gamma(1+a-b-d-e)}$$

$$\aleph_{U_{20};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(\text{d};\rho'_1,\cdots,\rho'_r),(e;\rho'_1,\cdots,\rho'_r),A:C\\ \cdots\\ \cdots\text{B : D}\end{array}\right)\quad(2.4)$$

where $U_{42}=p_i+4;q_i+2;\tau_i;R,\ U_{20}=p_i+2;q_i;\tau_i;R, \rho'_i=\dfrac{\rho_i}{2}, i=1,\cdots,r$

provided that the double series involved on the left hand side of (2.4) is absolutely convergent,$\rho_k>0, k=1,\cdots,r$

Proof

To prove (2.1), express the generalized multivariable Aleph-function in the sum on the left of (2.1) in terms of its Mellin-Barnes type contour integral (1.1), interchanging the order of integration ans summation, which is valid, as the series involved is finite and evaluating the inner double series with the help of the following known result of Sharma[1 (1976),p.187]

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(a)_{s+t}(a/2+1)_{s+t}(b)_{s+t}(c)_{s+t}(d)_s(e)_t}{(a/2)_{s+t}(1+a-b)_{s+t}(1+a-c)_{s+t}(1+a-d-e)_{s+t}s!t!}$$

$$=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d-e)\Gamma(1+a-b-c-d-e)}{\Gamma(1+a)\Gamma(1+a-b-c))\Gamma(1+a-b-d-e)\Gamma(1+a-c-d-e)}\quad(2.5)$$

where $Re(b+c+d+e-a)<1$

the proofs of (2.2) to (2.4) can be developed by similars methods to those given above by using (2.5) and the following result of Sharma[1 (1976),p.185].

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-)^{s+t}(a)_{s+t}(a/2+1)_{s+t}(b)_{s+t}(d)_s(e)_t}{(a/2)_{s+t}(1+a-b)_{s+t}(1+a-b)_{s+t}(1+a-d-e)_{s+t}s!t!}=\frac{\Gamma(1+a-b)\Gamma(1+a-d-e)}{\Gamma(1+a)\Gamma(1+a-b-d-e)} \quad (2.6)$$

with $Re(a/2-b-d-e)>1$

4. Generalized multivariable I-function

If $\tau_i=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=1$, the Aleph-function of several variables degenere to the generalizezd I-function of several variables. The finite four double sums have been derived in this section for generalized multivariable I-functions.

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(c)_{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}I_{U_{43};R:}^{m+4,\mathfrak{n}:V}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ z_r\end{matrix}\ \middle|\ \begin{matrix}(a+s+t;\rho_1,\cdots,\rho_r),(d+s;\rho'_1,\cdots,\rho'_r)\\ \cdots\\ (\tfrac{1}{2}+s+t;\rho'_1,\cdots,\rho'_r),\end{matrix}\right.$$

$$\left.\begin{matrix}(1+\tfrac{1}{2}+s+t;\rho'_1,\cdots,\rho'_r), & (e+t;\rho'_1,\cdots,\rho'_r), & A:C\\ \cdots & \cdots & \cdots\\ (1+a-b+s+t;\rho_1,\cdots,\rho_r), & (1+a-c+s+t;\rho_1,\cdots,\rho_r), & B:D\end{matrix}\right)=\frac{\Gamma(1+a-d-e)}{\Gamma(1+a-b-d-e)}$$

$$\frac{\Gamma(1+a-b-c-d-e)}{\Gamma(1+a-c-d-e)}\aleph_{U_{21};R:}^{m+2,\mathfrak{n}:V}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ z_r\end{matrix}\ \middle|\ \begin{matrix}(d;\rho'_1,\cdots,\rho'_r),(e;\rho'_1,\cdots,\rho'_r),A:C\\ \cdots\\ (1+a-b-c;\rho_1,\cdots,\rho_r),B:D\end{matrix}\right) \quad (3.1)$$

where the same notations and validity conditions that (2.1)

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(d)_s(e)_t}{s!t!}I_{U_{44};R:}^{m+4,\mathfrak{n}:V}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ z_r\end{matrix}\ \middle|\ \begin{matrix}(a+s+t;\rho_1,\cdots,\rho_r),(b+s+t;\rho'_1,\cdots,\rho'_r),\\ \cdots\\ (1+a-b+s+t;\rho'_1,\cdots,\rho'_r),(\tfrac{a}{2}+r+s;\rho'_1,\cdots,\rho'_r),\end{matrix}\right.$$

$$\left.\begin{matrix}(1+\tfrac{a}{2}+s+t;\rho'_1,\cdots,\rho'_r), & (c+s+t;\rho'_1,\cdots,\rho'_r), & A:C\\ \cdots & \cdots & \cdots\\ (1+a-c+s+t;\rho'_1,\cdots,\rho'_r), & (1+a-d-e+s+t;\rho_1,\cdots,\rho_r), & B:D\end{matrix}\right)=\frac{\Gamma(1+a-b-d-e)}{2\Gamma(1+a-c)}$$

$$I_{U_{22};R:}^{m+2,\mathfrak{n}:V}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ z_r\end{matrix}\ \middle|\ \begin{matrix}(b;\rho'_1,\cdots,\rho'_r),(c;\rho'_1,\cdots,\rho'_r),A:C\\ \cdots\\ (1+a-b-d-e;\rho'_1,\cdots,\rho'_r),(1+a-c-d-e;\rho'_1,\cdots,\rho'_r),B:D\end{matrix}\right) \quad (3.2)$$

where the same notations and validity conditions that (2.2)

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(d)_s(e)_s}{(1+a-b)_{s+t}s!t!}I_{U_{32};R:}^{m+3,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{matrix}\left| \begin{matrix} (\text{d+s+t };\rho_1,\cdots,\rho_r),(1+\frac{a}{2}+s+t;\rho_1',\cdots,\rho_r') \\ \cdots \\ (\frac{a}{2}\text{+s+t};\rho_1',\cdots,\rho_r'), \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (\text{b+s+t };\rho_1',\cdots,\rho_r'), \quad \mathrm{A:C} \\ \cdots \quad \cdots \\ (\text{1+a-d-e+s+t};\rho_1,\cdots,\rho_r),\mathrm{B:D} \end{matrix}\right)=\frac{\Gamma(1+a-b)}{\Gamma(1+a-b-d-e)}I_{U_{22};R:}^{m+2,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{matrix}\left| \begin{matrix} (\text{b};\rho_1,\cdots,\rho_r),A:C \\ \cdots \\ \mathrm{B:D} \end{matrix}\right.\right) \tag{3.3}$$

where the same notations and validity conditions that (2.3)

Formula 4

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}I_{U_{42};R:}^{m+4,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{matrix}\left| \begin{matrix} (\text{a+s+t };\rho_1,\cdots,\rho_r),(d+s;\rho_1',\cdots,\rho_r') \\ \cdots \\ (\frac{a}{2}\text{+s +t};\rho_1',\cdots,\rho_r'), \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (1+\frac{a}{2}\text{+s+t };\rho_1',\cdots,\rho_r'),(\text{e+s};\rho_1',\cdots,\rho_r'),\mathrm{A:C} \\ \cdots \quad \cdots \quad \cdots \\ (\text{1+a-b+s+t};\rho_1,\cdots,\rho_r), \quad \cdots, \quad \mathrm{B:D} \end{matrix}\right)=\frac{\Gamma(1+a-d-e)}{2\Gamma(1+a-b-d-e)}$$

$$I_{U_{20};R:}^{m+2,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{matrix}\left| \begin{matrix} (\text{d };\rho_1',\cdots,\rho_r'),(e;\rho_1',\cdots,\rho_r'),A:C \\ \cdots \\ \cdots \mathrm{B:D} \end{matrix}\right.\right) \tag{3.4}$$

where the same notations and validity conditions that (2.4)

4. Aleph-function of two variables

If $r=2$, we obtain the generalized Aleph-function of two variables and we have the following formulas

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(c)_{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}\aleph_{U_{43};R:}^{m+4,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{matrix}\left| \begin{matrix} (\text{a+s+t };\rho_1,\rho_2),(d+s;\rho_1',\rho_2') \\ \cdots \\ (\frac{1}{2}\text{+s +t};\rho_1',\rho_2'), \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (1+\frac{1}{2}\text{+s+t };\rho_1',\rho_2'), \quad (\text{e+t};\rho_1',\rho_2'), \quad \mathrm{A:C} \\ \cdots \quad \cdots \quad \cdots \\ (\text{1+a-b+s+t};\rho_1,\rho_2),(\text{1+a-c+s+t};\rho_1,\rho_2), \quad \mathrm{B:D} \end{matrix}\right)=\frac{\Gamma(1+a-d-e)}{\Gamma(1+a-b-d-e)}$$

$$\frac{\Gamma(1+a-b-c-d-e)}{\Gamma(1+a-c-d-e)}\aleph_{U_{21};R:}^{m+2,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_2 \end{matrix}\left| \begin{matrix} (\text{d };\rho_1',\rho_2'),(e;\rho_1',\rho_2'),A:C \\ \cdots \\ (\text{1+a-b-c };\rho_1,\rho_2),B:D \end{matrix}\right.\right) \tag{4.1}$$

where the same notations and validity conditions that (2.1)

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(d)_s(e)_t}{s!t!}\aleph_{U_{44};R:}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle|\begin{array}{c} (\text{a+s+t };\rho_1,\rho_2),(b+s+t;\rho'_1,\rho'_2), \\ \cdots \\ (\text{1+a-b+s +t};\rho'_1,\rho'_2),(\frac{a}{2}+r+s;\rho'_1,\rho'_2), \end{array}\right.$$

$$\left.\begin{array}{ccc} (1+\frac{a}{2}\text{+s+t };\rho'_1,\rho'_2), & (\text{c+s+t};\rho'_1,\rho'_2), & A:C \\ \cdots & \cdots & \cdots \\ (\text{1+a-c+s+t};\rho'_1,\rho'_2), & (\text{1+a-d-e+s+t};\rho_1,\rho_2), & B:D \end{array}\right)=\frac{\Gamma(1+a-b-d-e)}{2\Gamma(1+a-c)}$$

$$\aleph_{U_{22};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle|\begin{array}{c} (\text{b };\rho'_1,\rho'_2),(c;\rho'_1,\rho'_2),A:C \\ \cdots \\ (\text{1+a-b-d-e };\rho'_1,\rho'_2),(1+a-c-d-e;\rho'_1,\rho'_2),B:D \end{array}\right) \tag{4.2}$$

where the same notations and validity conditions that (2.2).

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(d)_s(e)_s}{(1+a-b)_{s+t}s!t!}\aleph_{U_{32};R:}^{m+3,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle|\begin{array}{c} (\text{d+s+t };\rho_1,\rho_2),(1+\frac{a}{2}+s+t;\rho'_1,\rho'_2) \\ \cdots \\ (\frac{a}{2}\text{+s+t};\rho'_1,\rho'_2), \end{array}\right.$$

$$\left.\begin{array}{cc} (\text{b+s+t };\rho'_1,\rho'_2), & A:C \\ \cdots & \cdots \\ (\text{1+a-d-e+s+t};\rho_1,\rho_2), & B:D \end{array}\right)=\frac{\Gamma(1+a-b)}{\Gamma(1+a-b-d-e)}\aleph_{U_{22};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle|\begin{array}{c} (\text{b};\rho_1,\rho_2),A:C \\ \cdots \\ B:D \end{array}\right) \tag{4.3}$$

where the same notations and validity conditions that (2.3)

Formula 4

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}\aleph_{U_{42};R:}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle|\begin{array}{cc} (\text{a+s+t };\rho_1,\rho_2),(d+s;\rho'_1,\rho'_2), & (\text{e+s};\rho'_1,\rho'_2), \\ \cdots & \cdots \\ (\frac{a}{2}\text{+s +t};\rho'_1,\rho'_2), & \cdots, \end{array}\right.$$

$$\left.\begin{array}{c} (1+\frac{a}{2}\text{+s+t };\rho'_1,\rho'_2),A:C \\ \cdots \quad \cdots \\ (\text{1+a-b+s+t};\rho_1,\rho_2),B:D \end{array}\right)=\frac{\Gamma(1+a-d-e)}{2\Gamma(1+a-b-d-e)}\aleph_{U_{20};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle|\begin{array}{c} (\text{d };\rho'_1,\rho'_2),(e;\rho'_1,\rho'_2),A:C \\ \cdots \\ \cdots B:D \end{array}\right) \tag{4.4}$$

where the same notations and validity conditions that (2.3)

5. I-function of two variables

If $\tau_i=\tau'_i=\tau''_i=1$, then the generalized Aleph-function of two variables degenere to generalized I-function of two variables and we obtain the following results.

Formula 1

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(c)_{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}\,I_{U_{43};R:}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\left|\begin{array}{c} (\text{a+s+t};\rho_1,\rho_2),(d+s;\rho_1',\rho_2') \\ \cdots \\ (\tfrac{1}{2}+\text{s}+\text{t};\rho_1',\rho_2'), \end{array}\right.\right.$$

$$\left.\begin{array}{ccc} (1+\tfrac{1}{2}+\text{s+t};\rho_1',\rho_2'), & (\text{e+t};\rho_1',\rho_2'), & \text{A}:\text{C} \\ \cdots & \cdots & \cdots \\ (1+\text{a-b+s+t};\rho_1,\rho_2), & (1+\text{a-c+s+t};\rho_1,\rho_2), & \text{B}:\text{D} \end{array}\right)=\frac{\Gamma(1+a-d-e)}{\Gamma(1+a-b-d-e)}$$

$$\frac{\Gamma(1+a-b-c-d-e)}{\Gamma(1+a-c-d-e)}I_{U_{21};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\left|\begin{array}{c} (\text{d};\rho_1',\rho_2'),(e;\rho_1',\rho_2'),A:C \\ \cdots \\ (1+\text{a-b-c};\rho_1,\rho_2),B:D \end{array}\right.\right) \qquad (5.1)$$

where the same notations and validity conditions that (2.1)

Formula 2

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(d)_s(e)_t}{s!t!}I_{U_{44};R:}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\left|\begin{array}{c} (\text{a+s+t};\rho_1,\rho_2),(b+s+t;\rho_1',\rho_2'), \\ \cdots \\ (1+\text{a-b+s+t};\rho_1',\rho_2'),(\tfrac{a}{2}+r+s;\rho_1',\rho_2'), \end{array}\right.\right.$$

$$\left.\begin{array}{ccc} (1+\tfrac{a}{2}+\text{s+t};\rho_1',\rho_2'), & (\text{c+s+t};\rho_1',\rho_2'), & \text{A}:\text{C} \\ \cdots & \cdots & \cdots \\ (1+\text{a-c+s+t};\rho_1',\rho_2'), & (1+\text{a-d-e+s+t};\rho_1,\rho_2), & \text{B}:\text{D} \end{array}\right)=\frac{\Gamma(1+a-b-d-e)}{2\Gamma(1+a-c)}$$

$$I_{U_{22};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\left|\begin{array}{c} (\text{b};\rho_1',\rho_2'),(c;\rho_1',\rho_2'),A:C \\ \cdots \\ (1+\text{a-b-d-e};\rho_1',\rho_2'),(1+a-c-d-e;\rho_1',\rho_2'),B:D \end{array}\right.\right) \qquad (5.2)$$

where the same notations and validity conditions that (2.2).

Formula 3

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(d)_s(e)_s}{(1+a-b)_{s+t}s!t!}I_{U_{32};R:}^{m+3,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\left|\begin{array}{c} (\text{d+s+t};\rho_1,\rho_2),(1+\tfrac{a}{2}+s+t;\rho_1',\rho_2') \\ \cdots \\ (\tfrac{a}{2}+\text{s+t};\rho_1',\rho_2'), \end{array}\right.\right.$$

$$\left.\begin{array}{cc} (\text{b+s+t};\rho_1',\rho_2'), & \text{A}:\text{C} \\ \cdots & \cdots \\ (1+\text{a-d-e+s+t};\rho_1,\rho_2), & \text{B}:\text{D} \end{array}\right)=\frac{\Gamma(1+a-b)}{\Gamma(1+a-b-d-e)}I_{U_{22};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\left|\begin{array}{c} (\text{b};\rho_1,\rho_2),A:C \\ \cdots \\ \text{B}:\text{D} \end{array}\right.\right) \qquad (5.3)$$

where the same notations and validity conditions that (2.3)

Formula 4

$$\sum_{s=0}^{m}\sum_{t=0}^{n}\frac{(-)^{s+t}(b)_{s+t}}{(1+a-d-e)_{s+t}s!t!}I_{U_{42};R:}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(a+s+t;\rho_1,\rho_2),(d+s;\rho_1',\rho_2'),\ (e+s;\rho_1',\rho_2'),\\ \cdots\qquad\cdots\\ (\frac{a}{2}+s+t;\rho_1',\rho_2'),\qquad \cdots,\end{array}\right.$$

$$\left.\begin{array}{c}(1+\frac{a}{2}+s+t;\rho_1',\rho_2'),\mathrm{A}:\mathrm{C}\\ \cdots\qquad\cdots\\ (1+a-b+s+t;\rho_1,\rho_2),\mathrm{B}:\mathrm{D}\end{array}\right)=\frac{\Gamma(1+a-d-e)}{2\Gamma(1+a-b-d-e)}I_{U_{20};R:}^{m+2,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(d;\rho_1',\rho_2'),(e;\rho_1',\rho_2'),A:C\\ \cdots\\ \cdots\mathrm{B}:\mathrm{D}\end{array}\right)\quad(5.4)$$

6. Conclusion

The generalized aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function defined by Sharma et al [2] ,the Aleph-function of two variables defined by [4], the I-function of two variables defined by Sharma[3] , Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel func[]tion of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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