On some multidimensional integral transforms of multivariable Aleph-functions I

$F.Y.\ AYANT^1$

1 Teacher in High School , France

ABSTRACT
In this present document, we obtain multidimensional Laplace transforms and Whittaker transforms of Aleph-function of several variables. During the course of finding, we establish several particular cases.

Keywords :Multivariable Aleph-function, multidimensional Laplace transforms , multidimentional Whittaker transforms . Multivariable I-function, Aleph-function of two variables.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [5] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} \mathrm{y}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r \end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\ldots\ldots\ldots\ldots\ldots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$

$$\left.\begin{array}{l} [(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ [(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

For more details, see Ayant [1].

The reals numbers τ_i are positives for $i=1,\cdots,R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1,\cdots,R^{(k)}$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)} \tag{1.2}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(y_1,\cdots,y_r) = 0(\,|y_1|^{\alpha_1}\ldots|y_r|^{\alpha_r}\,)\,, max(\,|y_1|\ldots|y_r|\,) \rightarrow 0$$

$$\aleph(y_1,\cdots,y_r) = 0(\,|y_1|^{\beta_1}\ldots|y_r|^{\beta_r}\,)\,, min(\,|y_1|\ldots|y_r|\,) \rightarrow \infty$$

where, with $k = 1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \;;\; V = m_1, n_1; \cdots; m_r, n_r \tag{1.3}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.4}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.5}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.6}$$

$$C_1 = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \tag{1.7}$$

$$C_r = \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.8}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.9}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r) = \aleph_{U:W}^{0,\mathrm{n}:V}\left(\begin{array}{c|c} z_1 & \mathrm{A} : \mathrm{C}_1 : \cdots : C_r \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ z_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.10}$$

2. Multivariable Laplace Transform

Chandel [2] introduced the multivariable Laplace transform

$$L_{a_1,\cdots,a_n}^{\lambda,\mu}\{\} = \frac{\Gamma(a_1+\cdots+a_n)\lambda^{a_1+\cdots+a_n+\mu}}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(a_1+\cdots+a_n+\mu)}\int_0^\infty\cdots\int_0^\infty e^{-\lambda\sum_{i=1}^n x_i}\,(\sum_{i=1}^n x_i)^\mu$$

$$x_1^{a_1-1}\cdots x_n^{a_n-1}\{\}\,\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{2.1}$$

Where $Re(a_i) > 0, j = 1,\cdots,n, Re(\lambda) > 0$ and $Re(a_1+\cdots+a_n+\mu) > 0$

Here we give following additional application of the above operator

Let $X = \sum_{i=1}^n x_i\,, C_1\cdots, C_n$ are defined by (1.7) and (1.8), we have

$$L_{a_1,\cdots,a_n}^{\lambda,\mu}\{\aleph(u_1x^{\sigma_1}X^{v_1},\cdots,u_nx^{\sigma_n}X^{v_n})\} = \frac{\Gamma(a_1+\cdots+a_n)}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(a_1+\cdots+a_n+\mu)}$$

$$\aleph^{0,\mathfrak{n}+1:V+1}_{U_{11}:W+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\sigma_1+v_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\sigma_n+v_n} \end{array}\middle| \begin{array}{c} (1\text{-}\sum_{i=1}^n a_i-\mu:\sigma_1+v_1,\cdots,\sigma_n+v_n),A:(1\text{-a}_1;\sigma_1),C_1:\cdots: \\ \cdots \qquad \cdots \\ (1\text{-}\sum_{i=1}^n a_i:\sigma_1,\cdots,\sigma_n),B: \qquad \cdots\cdots \end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-a}_n;\sigma_n),C_n \\ \cdots \\ \mathrm{D}\end{array}\right) \tag{2.2}$$

where $U_{11}=p_i+1,q_i+1,\tau_i;R\,;V+1=m_1,n_1+1;\cdots;m_r,n_r+1$

and $W+1=p_{i^{(1)}}+1,q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}}+1,q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}$

Provided $Re(a_i)>0,j=1,\cdots,n,Re(\lambda)>0$ and $Re(a_1+\cdots+a_n+\mu)>0$

and $|argu_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.2) ; $\sigma_i,v_i>0$

Proof of (2.2) : Let $M=\frac{1}{(2\pi\omega)^n}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_n)\prod_{k=1}^n\theta_k(s_k)$, we have

$$L^{\lambda,\mu}_{a_1,\cdots,a_n}\{\aleph(u_1x^{\sigma_1}X^{v_1},\cdots,u_nx^{\sigma_n}X^{v_n})\}=L^{\lambda,\mu}_{a_1,\cdots,a_n}\{M\{\prod_{k=1}^n\left(u_ix^{\sigma_i}X^{v_i}\right)^{s_i}\}\,\mathrm{d}s_1\cdots\mathrm{d}s_n\}$$

We interchange the order of integrations (which is permissible under the conditions stated), we obtain

$$M\{L^{\lambda,\mu}_{a_1,\cdots,a_n}\{\prod_{k=1}^n\left(u_ix^{\sigma_i}X^{v_i}\right)^{s_i}\}\}\mathrm{d}s_1\cdots\mathrm{d}s_n$$

Now evaluating the inner multiple Laplace integral,see [1], after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result

3. Generalized Whittaker transforms

Chandel and Dwivedi [3] and [4] introduced the multivariable Whittaker transform

$$W^{\lambda,\mu,\upsilon}_{a_1,\cdots,a_n,\sigma}\{\}=\frac{\Gamma(\sum_{i=1}^n a_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n a_i)\lambda^{\sum_{i=1}^n a_i}}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(\sigma+\sum_{i=1}^n a_i+1/2\pm\upsilon)}$$

$$\int_0^\infty\cdots\int_0^\infty x_1^{a_1-1}\cdots x_n^{a_n-1}e^{-\lambda(\sum_{i=1}^n x_i)/2}\left(\sum_{i=1}^n x_i\right)^\sigma W_{\mu,\upsilon}\left(\lambda\sum_{i=1}^n x_i\right)\{\}\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{3.1}$$

where $Re(a_i)>0,j=1,\cdots,n,Re(\lambda)>0$ and $Re(\sigma+1/2+a_1+\cdots+a_n\pm\upsilon)>0$

Here we give following additional application of the above operator

Let $X=\sum_{i=1}^n x_i\,,C_1\cdots,C_n$ are defined by (1.7) and (1.8), we have

$$W^{\lambda,\mu,\upsilon}_{a_1,\cdots,a_n,\sigma}\{\aleph(u_1x^{\sigma_1}X^{\rho_1},\cdots,u_nx^{\sigma_n}X^{\rho_n})\}=\frac{\Gamma(\sum_{i=1}^n a_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n a_i)}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(\sigma+\sum_{i=1}^n a_i+1/2\pm\upsilon)}$$

$$\aleph_{U_{22}:W+1}^{0,\mathfrak{n}+2:V+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\sigma_1+\rho_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\sigma_n+\rho_n} \end{array}\middle| \begin{array}{c} (1/2\text{-}\sum_{i=1}^n a_i-(\sigma\pm\upsilon):\sigma_1+\rho_1,\cdots,\sigma_n+\rho_n), \\ \cdots \\ (\mu-\sigma-\sum_{i=1}^n a_i:\sigma_1+\rho_1,\cdots,\sigma_n+\rho_n), \end{array}\right.$$

$$\left.\begin{array}{ccc} \mathrm{A}: & (1\text{-a}_1;\sigma_1),C_1:\cdots:(1\text{-a}_n;\sigma_n),C_n \\ \cdots & \cdots & \cdots \\ (1\text{-}\sum_{i=1}^n a_i:\sigma_1,\cdots,\sigma_n),B: & \cdots\cdots & \mathrm{D} \end{array}\right) \tag{3.2}$$

where $U_{22}=p_i+2,q_i+2,\tau_i;R\,;V+1=m_1,n_1+1;\cdots;m_r,n_r+1$

and $W+1=p_{i^{(1)}}+1,q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}}+1,q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}$

Provided $Re(a_i)>0,j=1,\cdots,n,Re(\lambda)>0$ and $Re(\sigma+1/2+a_1+\cdots+a_n\pm\upsilon)>0$

and $|argu_k|<\dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.2) $\sigma_i,\rho_i>0$

Proof of (3.2) : Let $M=\dfrac{1}{(2\pi\omega)^n}\displaystyle\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_n)\prod_{k=1}^n\theta_k(s_k)$, we have

$$W_{a_1,\cdots,a_n,\sigma}^{\lambda,\mu,\upsilon}\{\aleph(u_1x^{\sigma_1}X^{\upsilon_1},\cdots,u_nx^{\sigma_n}X^{\upsilon_n})\}=W_{a_1,\cdots,a_n,\sigma}^{\lambda,\mu,\upsilon}\{M\{\prod_{k=1}^n(u_ix^{\sigma_i}X^{\rho_i})^{s_i}\}\,\mathrm{d}s_1\cdots\mathrm{d}s_n\}$$

We interchange the order of integrations (which is permissible under the conditions stated), we obtain

$$M\{W_{a_1,\cdots,a_n,\sigma}^{\lambda,\mu,\upsilon}\{\prod_{k=1}^n(u_ix^{\sigma_i}X^{\rho_i})^{s_i}\}\}\mathrm{d}s_1\cdots\mathrm{d}s_n$$

Now evaluating the inner multiple Whittaker integral, see [2], after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result

Special cases of (3.2)

a) For $\sigma_1=\cdots\sigma_n=0$, we derive from (3.2)

$$W_{a_1,\cdots,a_n,\sigma}^{\lambda,\mu,\upsilon}\{\aleph(u_1X^{\rho_1},\cdots,u_nX^{\rho_n})\}=\frac{\Gamma(\sum_{i=1}^n a_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n a_i)}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(\sigma+\sum_{i=1}^n a_i+1/2\pm\upsilon)}$$

$$\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\rho_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\rho_n} \end{array}\middle| \begin{array}{cc} (1/2\text{-}\sum_{i=1}^n a_i-(\sigma\pm\upsilon):\rho_1,\cdots,\rho_n), & , \\ \cdots & \cdots \\ (\mu-\sigma-\sum_{i=1}^n a_i:\rho_1,\cdots,\rho_n), & (1\text{-}\sum_{i=1}^n a_i:\sigma_1,\cdots,\sigma_n), \end{array}\right.$$

$$\left.\begin{array}{c} \mathrm{A}:\mathrm{C}_1:\cdots:\mathrm{C}_n \\ \cdots\ \ \cdots\ \ \cdots \\ \mathrm{B}:\cdots\cdots\ \mathrm{D} \end{array}\right) \tag{3.3}$$

where $U_{21}=p_i+2,q_i+1,\tau_i;R\,;$ Provided $Re(a_i)>0,j=1,\cdots,n,Re(\lambda)>0$ and

$Re(\sigma+1/2+a_1+\cdots+a_n\pm\upsilon)>0$

and $|argu_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.2) $\rho_i > 0$

a) For $\rho_1 = \cdots \rho_n = 0$, we derive from (3.2)

$$W^{\lambda,\mu,\upsilon}_{a_1,\cdots,a_n,\sigma}\{\aleph(u_1x^{\sigma_1},\cdots,u_nx^{\sigma_n})\} = \frac{\Gamma(\sum_{i=1}^n a_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n a_i)}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(\sigma+\sum_{i=1}^n a_i+1/2\pm\upsilon)}$$

$$\aleph^{0,\mathfrak{n}+2:V+1}_{U_{22}:W+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\sigma_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\sigma_n} \end{array}\middle| \begin{array}{cc} (1/2\text{-}\sum_{i=1}^n a_i-(\sigma\pm\upsilon):\sigma_1,\cdots,\sigma_n), & \mathrm{A}: \\ \cdots & \cdots \\ (\mu-\sigma-\sum_{i=1}^n a_i:\sigma_1,\cdots,\sigma_n), & (1\text{-}\sum_{i=1}^n a_i:\sigma_1,\cdots,\sigma_n),B: \end{array}\right.$$

$$\left.\begin{array}{cc} (1\text{-a}_1;\sigma_1),C_1:\cdots:(1\text{-a}_n;\sigma_n),C_n \\ \cdots \quad\quad \cdots \\ \cdots\cdots \quad\quad \mathrm{D} \end{array}\right) \tag{3.4}$$

where $U_{22}=p_i+2,q_i+2,\tau_i;R\,;V+1=m_1,n_1+1;\cdots;m_r,n_r+1$

and $W+1=p_{i^{(1)}}+1,q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}}+1,q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}$

Provided $Re(a_i)>0,j=1,\cdots,n,Re(\lambda)>0$ and $Re(\sigma+1/2+a_1+\cdots+a_n\pm\upsilon)>0$

and $|argu_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.2) $\sigma_i > 0$

4. Other Multidimentional Whittaker transform

Chandel and Dwivedi [4] introduced and studied the multidimensional Whittaker transform

$$T^{\lambda,\mu,\upsilon}_{\beta_1,\cdots,\beta_n,\sigma}\{\} = \frac{K\lambda^{\sigma+\sum_{i=1}^n\beta_j}\Gamma(\sum_{i=1}^n\beta_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n\beta_i)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\sigma+\sum_{i=1}^n\beta_i+1/2\pm\upsilon)}\int_0^\infty\cdots\int_0^\infty$$

$$\prod_{j=1}^n(\sum_{i=1}^n a_i^jx_i)^{\beta_j-1}e^{-\lambda(\sum_{j,i=1}^n a_i^jx_i)/2}\sum_{j,i=1}^n(a_i^jx_i)^\sigma W_{\mu,\upsilon}(\lambda\sum_{j,i=1}^n a_i^jx_i)\{\}\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{4.1}$$

where $Re(\beta_i)>0,j=1,\cdots,n,Re(\lambda)>0$ and $Re(\sigma+1/2+\beta_1+\cdots+\beta_n\pm\upsilon)>0,$

$Re(\sigma+1+\beta_1+\cdots+\beta_n-\mu)>0$ and $K=\begin{vmatrix} a_1^1 & \cdots & a_n^1 \\ \cdots & & \\ a_1^n & \cdots & a_n^n \end{vmatrix}\neq 0$

Let $X_1=\sum_{i=1}^n a_i^1x_i,\cdots,X_n=\sum_{i=1}^n a_i^nx_i$

$$T^{\lambda,\mu,\upsilon}_{\beta_1,\cdots,\beta_n,\sigma}\{\aleph(u_1X_1^{\eta_1},\cdots,u_nX_n^{\eta_n})\} = \frac{K\lambda^{\sigma+\sum_{i=1}^n\beta_j}\Gamma(\sum_{i=1}^n\beta_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n\beta_i)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\sigma+\sum_{i=1}^n\beta_i+1/2\pm\upsilon)}$$

$$\aleph^{0,\mathfrak{n}+2:V+1}_{U_{22}:W+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\eta_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\eta_n} \end{array}\middle| \begin{array}{cc} (1/2\text{-}(\sigma\pm\upsilon)-\sum_{i=1}^n\beta_i:\eta_1,\cdots,\eta_n), & \mathrm{A}: \\ \cdots & \cdots \\ (\mu-\sigma-\sum_{i=1}^n\beta_i:\eta_1,\cdots,\eta_n), & (1\text{-}\sum_{i=1}^n\beta_i:\eta_1,\cdots,\eta_n),B: \end{array}\right.$$

$$\left.\begin{matrix} (1\text{-}\beta_1;\sigma_1),C_1:\cdots:(1\text{-}\beta_n;\sigma_n),C_n \\ \cdots \qquad \cdots \\ \cdots\cdots \qquad \text{D} \end{matrix}\right) \tag{4.2}$$

where $U_{22}=p_i+2,q_i+2,\tau_i;R\,;V+1=m_1,n_1+1;\cdots;m_r,n_r+1$

and $W+1=p_{i^{(1)}}+1,q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}}+1,q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}$

Provided $Re(a_i)>0, j=1,\cdots,n, Re(\lambda)>0$ and $Re(\sigma+1/2+a_1+\cdots+a_n\pm\upsilon)>0$

$Re(\sigma+1+\beta_1+\cdots+\beta_n-\mu)>0$ and $K=\begin{vmatrix} a_1^1 & \cdots & a_n^1 \\ \cdots & & \\ a_1^n & \cdots & a_n^n \end{vmatrix}\neq 0$

and $|argu_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.2) $\sigma_i>0$

Proof of (4.2) : Let $M=\frac{1}{(2\pi\omega)^n}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_n)\prod_{k=1}^{n}\theta_k(s_k)$, we have

$$T^{\lambda,\mu,\upsilon}_{\beta_1,\cdots,\beta_n,\sigma}\{\aleph(u_1x^{\sigma_1}X^{\upsilon_1},\cdots,u_nx^{\sigma_n}X^{\upsilon_n})\}=T^{\lambda,\mu,\upsilon}_{\beta_1,\cdots,\beta_n,\sigma}\{M\{\prod_{k=1}^{n}\left(u_ix^{\sigma_i}X^{\eta_i}\right)^{s_i}\}\,\mathrm{d}s_1\cdots\mathrm{d}s_n\}$$

We interchange the order of integrations (which is permissible under the conditions stated), we obtain

$$M\{T^{\lambda,\mu,\upsilon}_{\beta_1,\cdots,\beta_n,\sigma}\{\prod_{k=1}^{n}\left(u_ix^{\sigma_i}X^{\eta_i}\right)^{s_i}\}\}\mathrm{d}s_1\cdots\mathrm{d}s_n$$

Now evaluating the inner multiple Whittaker integral, see [3], after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result.

Let $X_1=\sum_{i=1}^{n}a_i^1x_i,\cdots,X_n=\sum_{i=1}^{n}a_i^nx_i$ and $X_{nn}=\sum_{j,i=1}^{n}a_i^jx_i$, we have

$$T^{\lambda,\mu,\upsilon}_{\beta_1,\cdots,\beta_n,\sigma}\{\aleph(u_1X_1^{\eta_1}X_{nn}^{\zeta_1},\cdots,u_nX_n^{\eta_n}X_{nn}^{\zeta_n})\}=\frac{K\lambda^{\sigma+\sum_{i=1}^{n}\beta_j}\Gamma(\sum_{i=1}^{n}\beta_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^{n}\beta_i)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\sigma+\sum_{i=1}^{n}\beta_i+1/2\pm\upsilon)}$$

$$\aleph^{0,\mathfrak{n}+2:V+1}_{U_{22}:W+1}\left(\begin{matrix} \mathrm{u}_1/\lambda^{\eta_1+\zeta_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\eta_n+\zeta_n} \end{matrix}\ \middle|\ \begin{matrix} (1/2\text{-}(\sigma\pm\upsilon)-\sum_{i=1}^{n}\beta_i:\eta_1+\zeta_1,\cdots,\eta_n+\zeta_n), \\ \cdots \\ (\mu-\sigma-\sum_{i=1}^{n}\beta_i:\eta_1+\zeta_1,\cdots,\eta_n+\zeta_n), \end{matrix}\right.$$

$$\left.\begin{matrix} \mathrm{A}: & (1\text{-}\beta_1;\sigma_1),C_1:\cdots:(1\text{-}\beta_n;\sigma_n):C_n \\ \cdots & \cdots \qquad \cdots \\ (1\text{-}\sum_{i=1}^{n}\beta_i:\eta_1,\cdots,\eta_n),B: & \cdots\cdots \qquad \text{D} \end{matrix}\right) \tag{4.3}$$

where $U_{22}=p_i+2,q_i+2,\tau_i;R\,;V+1=m_1,n_1+1;\cdots;m_r,n_r+1$

and $W+1=p_{i^{(1)}}+1,q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}}+1,q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}$

Provided $Re(a_i)>0, j=1,\cdots,n, Re(\lambda)>0$ and $Re(\sigma+1/2+a_1+\cdots+a_n\pm\upsilon)>0$

$Re(\sigma+1+\beta_1+\cdots+\beta_n-\mu)>0$ and $K=\begin{vmatrix} a_1^1 & \cdots & a_n^1 \\ \cdots & & \\ a_1^n & \cdots & a_n^n \end{vmatrix} \neq 0$

and $|arg u_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5) $\sigma_i > 0$ .To prove (4.3), the method is similar that (4.2).

5. Multivariable I-function

In these section, we get several multidimensional transforms concerning the multivariable I-function defined by Sharma et al [5] Let $\tau_i=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=1$, we have the following relations.

a) $L^{\lambda,\mu}_{a_1,\cdots,a_n}\{I(u_1x^{\sigma_1}X^{v_1},\cdots,u_nx^{\sigma_n}X^{v_n})\} = \frac{\Gamma(a_1+\cdots+a_n)}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(a_1+\cdots+a_n+\mu)}$

$$I^{0,\mathrm{n}+1:V+1}_{U_{11}:W+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\sigma_1+v_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\sigma_n+v_n} \end{array}\middle| \begin{array}{l} (1\text{-}\sum_{i=1}^n a_i-\mu:\sigma_1+v_1,\cdots,\sigma_n+v_n),A':(1\text{-a}_1;\sigma_1),C'_1:\cdots:(1\text{-a}_n;\sigma_n),C'_n \\ \cdots \\ (1\text{-}\sum_{i=1}^n a_i:\sigma_1,\cdots,\sigma_n),B': \quad \cdots\cdots \quad \cdots \quad \text{D'} \end{array}\right) \tag{5.1}$$

where $L^{\lambda,\mu}_{a_1,\cdots,a_n}\{\}$ is the multivariable Laplace transform defined by Chandel [1] and $X=\sum_{i=1}^n x_i$

which holds true under the same conditions from (2.2)

b) $W^{\lambda,\mu,\upsilon}_{a_1,\cdots,a_n,\sigma}\{I(u_1x^{\sigma_1}X^{\rho_1},\cdots,u_nx^{\sigma_n}X^{\rho_n})\} = \frac{\Gamma(\sum_{i=1}^n a_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n a_i)}{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(\sigma+\sum_{i=1}^n a_i+1/2\pm\upsilon)}$

$$I^{0,\mathrm{n}+2:V+1}_{U_{22}:W+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\sigma_1+\rho_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\sigma_n+\rho_n} \end{array}\middle| \begin{array}{l} (1/2\text{-}\sum_{i=1}^n a_i-(\sigma\pm\upsilon):\sigma_1+\rho_1,\cdots,\sigma_n+\rho_n), \\ \cdots \\ (\mu-\sigma+\sum_{i=1}^n a_i:\sigma_1+\rho_1,\cdots,\sigma_n+\rho_n), \end{array}\right.$$

$$\left.\begin{array}{lll} \text{A'}: & (1\text{-a}_1;\sigma_1),C'_1:\cdots:(1\text{-a}_n;\sigma_n),C'_n & \\ \cdots & \cdots & \cdots \\ (1\text{-}\sum_{i=1}^n a_i:\sigma_1,\cdots,\sigma_n),B': & \cdots\cdots & \text{D'} \end{array}\right) \tag{5.2}$$

where $W^{\lambda,\mu,\upsilon}_{a_1,\cdots,a_n,\sigma}\{\}$ is multivariable Whittaker transform defined by Chandel and Dwivedi [2] and [3] and

$X=\sum_{i=1}^n x_i$ which holds true under the same conditions from (3.2)

c) $T^{\lambda,\mu,\upsilon}_{\beta_1,\cdots,\beta_n,\sigma}\{I(u_1X_1^{\eta_1},\cdots,u_nX_n^{\eta_n})\} = \frac{K\lambda^{\sigma+\sum_{i=1}^n\beta_j}\Gamma(\sum_{i=1}^n\beta_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^n\beta_i)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\sigma+\sum_{i=1}^n\beta_i+1/2\pm\upsilon)}$

$$I_{U_{22}:W+1}^{0,\mathfrak{n}+2:V+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\eta_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\eta_n} \end{array}\middle| \begin{array}{cc} (1/2\text{-}(\sigma\pm\upsilon)-\sum_{i=1}^{n}\beta_i:\eta_1,\cdots,\eta_n), & \mathrm{A}: \\ \cdots & \cdots \\ (\mu-\sigma-\sum_{i=1}^{n}\beta_i):\eta_1,\cdots,\eta_n), & (1\text{-}\sum_{i=1}^{n}\beta_i:\eta_1,\cdots,\eta_n),B: \end{array}\right.$$

$$\left.\begin{array}{cc} (1\text{-}\beta_1;\sigma_1),C_1:\cdots:(1\text{-}\beta_n;\sigma_n):C_n \\ \cdots \quad \cdots \\ \cdots\cdots \quad \mathrm{D} \end{array}\right) \tag{5.3}$$

where $T_{\beta_1,\cdots,\beta_n,\sigma}^{\lambda,\mu,\upsilon}\{\}$ is multivariable Whittaker transform defined by Chandel and Dwivedi [4], and

$X_1=\sum_{i=1}^{n}a_i^1x_i,\cdots,X_n=\sum_{i=1}^{n}a_i^nx_i$ **which holds true under the same conditions from (3.2)**

d) $T_{\beta_1,\cdots,\beta_n,\sigma}^{\lambda,\mu,\upsilon}\{I(u_1X_1^{\eta_1}X_{nn}^{\zeta_1},\cdots,u_nX_n^{\eta_n}X_{nn}^{\zeta_n})\}=\dfrac{K\lambda^{\sigma+\sum_{i=1}^{n}\beta_j}\Gamma(\sum_{i=1}^{n}\beta_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^{n}\beta_i)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\sigma+\sum_{i=1}^{n}\beta_i+1/2\pm\upsilon)}$

$$I_{U_{22}:W+1}^{0,\mathfrak{n}+2:V+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\eta_1+\zeta_1} \\ \cdot \\ \cdot \\ \mathrm{u}_n/\lambda^{\eta_n+\zeta_n} \end{array}\middle| \begin{array}{c} (1/2\text{-}(\sigma\pm\upsilon)-\sum_{i=1}^{n}\beta_i:\eta_1+\zeta_1,\cdots,\eta_n+\zeta_n), \\ \cdots \\ (\mu-\sigma-\sum_{i=1}^{n}\beta_i):\eta_1+\zeta_1,\cdots,\eta_n+\zeta_n), \end{array}\right.$$

$$\left.\begin{array}{ccc} \mathrm{A}: & (1\text{-}\beta_1;\sigma_1),C_1:\cdots:(1\text{-}\beta_n;\sigma_n):C_n \\ \cdots & \cdots \quad \cdots \\ (1\text{-}\sum_{i=1}^{n}\beta_i:\eta_1,\cdots,\eta_n),B: & \cdots\cdots \quad \mathrm{D} \end{array}\right) \tag{5.4}$$

where $W_{a_1,\cdots,a_n,\sigma}^{\lambda,\mu,\upsilon}\{\}$ is multivariable Whittaker transform defined by Chandel and Dwivedi [3] and [4] and

$X_1=\sum_{i=1}^{n}a_i^1x_i,\cdots,X_n=\sum_{i=1}^{n}a_i^nx_i$ and Let $X_{nn}=\sum_{j,i=1}^{n}a_i^jx_i$ **which holds true under the same conditions from (4.2)**

6. Aleph-function of two variables

In these section, we get several multidimensional transforms concerning oncerning the Aleph-function of two variables defined by K. Sharma [6].

a) $L_{a_1,a_2}^{\lambda,\mu}\{\aleph(u_1x^{\sigma_1}X^{\upsilon_1},u_2x^{\sigma_2}X^{\upsilon_2})\}=\dfrac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_1+a_2+\mu)}\aleph_{U_{11}:W+1}^{0,\mathfrak{n}+1:V+1}\left(\begin{array}{c} \mathrm{u}_1/\lambda^{\sigma_1+\upsilon_1} \\ \cdot \\ \cdot \\ \mathrm{u}_2/\lambda^{\sigma_2+\upsilon_2} \end{array}\middle|\right.$

$$\left.\begin{array}{ccc} (1\text{-}\sum_{i=1}^{2}a_i-\mu:\sigma_1+\upsilon_1,\sigma_2+\upsilon_2),A'':(1\text{-a}_1;\sigma_1),C_1'':(1\text{-a}_2;\sigma_2),C_2'' \\ \cdots \quad \cdots \quad \cdots \\ (1\text{-}\sum_{i=1}^{2}a_i:\sigma_1,\sigma_2),B'': \quad \cdots\cdots \quad \mathrm{D''} \end{array}\right) \tag{6.1}$$

where $L^{\lambda,\mu}_{a_1,a_2}\{\}$ is the doubleLaplace transform and $X=\sum_{i=1}^{2}x_i$ which holds true under the same conditions from (2.2) with $n=2$

b) $$W^{\lambda,\mu,\upsilon}_{a_1,a_2,\sigma}\{\aleph(u_1x^{\sigma_1}X^{\rho_1},u_2x^{\sigma_2}X^{\rho_2})\}=\frac{\Gamma(\sum_{i=1}^2 a_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^2 a_i)}{\Gamma(a_1)\Gamma(a_2)\Gamma(\sigma+\sum_{i=1}^2 a_i+1/2\pm\upsilon)}$$

$$\aleph^{0,\mathfrak{n}+2:V+1}_{U_2:W+1}\left(\begin{array}{c}\mathrm{u}_1/\lambda^{\sigma_1+\rho_1}\\ \cdot\\ \cdot\\ \mathrm{u}_2/\lambda^{\sigma_2+\rho_2}\end{array}\middle|\begin{array}{l}(1/2\text{-}\sum_{i=1}^2 a_i-(\sigma\pm\upsilon):\sigma_1+\rho_1,\sigma_2+\rho_2),\qquad \mathrm{A}":\\ \qquad\cdots\qquad\qquad\cdots\\ (\mu-\sigma+\sum_{i=1}^2 a_i:\sigma_1+\rho_1,\sigma_2+\rho_2),\quad(1\text{-}\sum_{i=1}^2 a_i:\sigma_1,\sigma_2),B'':\end{array}\right.$$

$$\left.\begin{array}{l}(1\text{-}\mathrm{a}_1;\sigma_1),C_1'':(1\text{-}\mathrm{a}_n;\sigma_2),C_2''\\ \quad\cdots\qquad\qquad\cdots\\ \cdots\cdots\qquad\qquad \mathrm{D}"\end{array}\right) \tag{6.2}$$

where $W^{\lambda,\mu,\upsilon}_{a_1,a_2,\sigma}\{\}$ double Whittaker transform and $X=\sum_{i=1}^{2}x_i$ which holds true under the same conditions from (3.2) with n =2

c) $$T^{\lambda,\mu,\upsilon}_{\beta_1,\beta_2,\sigma}\{\aleph(u_1X_1^{\eta_1},u_2X_2^{\eta_2})\}=\frac{k\lambda^{\sigma+\sum_{i=1}^2\beta_j}\Gamma(\sum_{i=1}^2\beta_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^2\beta_i)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\sigma+\sum_{i=1}^2\beta_i+1/2\pm\upsilon)}$$

$$\aleph^{0,\mathfrak{n}+2:V+1}_{U_2:W+1}\left(\begin{array}{c}\mathrm{u}_1/\lambda^{\eta_1}\\ \cdot\\ \cdot\\ \mathrm{u}_2/\lambda^{\eta_2}\end{array}\middle|\begin{array}{l}(1/2\text{-}(\sigma\pm\upsilon)-\sum_{i=1}^2\beta_i:\eta_1,\eta_2),\qquad \mathrm{A}":\\ \qquad\cdots\qquad\qquad\cdots\\ (\mu-\sigma-\sum_{i=1}^2\beta_i):\eta_1,\eta_2),\quad(1\text{-}\sum_{i=1}^2\beta_i:\eta_1,\eta_2),B'':\end{array}\right.$$

$$\left.\begin{array}{l}(1\text{-}\beta_1;\sigma_1)C_1'':(1\text{-}\beta_2;\sigma_2),C_2''\\ \quad\cdots\qquad\qquad\cdots\\ \cdots\cdots\qquad\qquad \mathrm{D}"\end{array}\right) \tag{6.3}$$

where $T^{\lambda,\mu,\upsilon}_{\beta_1,\beta_2,\sigma}\{\}$ is doubleWhittaker transform and

$X_1=\sum_{i=1}^{n}a_i^1x_i, X_2=\sum_{i=1}^{2}a_i^2x_i$ which holds true under the same conditions from (4.2) with $n=2$

d) $$T^{\lambda,\mu,\upsilon}_{\beta_1,\beta_2,\sigma}\{\aleph(u_1X_1^{\eta_1}X_{22}^{\zeta_1},u_2X_2^{\eta_2}X_{22}^{\zeta_2})\}=\frac{k\lambda^{\sigma+\sum_{i=1}^2\beta_j}\Gamma(\sum_{i=1}^2\beta_i+\sigma+1-\mu)\Gamma(\sum_{i=1}^2\beta_i)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\sigma+\sum_{i=1}^2\beta_i+1/2\pm\upsilon)}$$

$$\aleph^{0,\mathfrak{n}+2:V+1}_{U_2:W+1}\left(\begin{array}{c}\mathrm{u}_1/\lambda^{\eta_1+\zeta_1}\\ \cdot\\ \cdot\\ \mathrm{u}_2/\lambda^{\eta_2+\zeta_2}\end{array}\middle|\begin{array}{l}(1/2\text{-}(\sigma\pm\upsilon)-\sum_{i=1}^2\beta_i:\eta_1+\zeta_1,\eta_2+\zeta_2),\\ \qquad\cdots\\ (\mu-\sigma-\sum_{i=1}^2\beta_i):\eta_1+\zeta_1,\eta_2+\zeta_2),\end{array}\right.$$

$$\begin{matrix} \text{A''}: & (1\text{-}\beta_1;\sigma_1)C_1'' : (1\text{-}\beta_2;\sigma_2), C_2'' \\ \cdots & \cdots \quad\quad \cdots \\ (1\text{-}\sum_{i=1}^{2}\beta_i : \eta_1,\eta_2), B'' : & \cdots\cdots \quad\quad \text{D''} \end{matrix}\Bigg) \tag{6.4}$$

where $W_{a_1,a_2,\sigma}^{\lambda,\mu,\upsilon}\{\}$ is the double Whittaker transform and $X_1 = \sum_{i=1}^{2} a_i^1 x_i, X_2 = \sum_{i=1}^{2} a_i^2 x_i$ and $X_{22} = \sum_{j,i=1}^{2} a_i^j x_i$ **which holds true under the same conditions from (4.2) with** $n = 2$

6. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as , multivariable H-function , defined by Srivastava et al [7] , the Aleph-function of two variables defined by K.sharma [6].

REFERENCES

[1] Ayant F.Y. An integral associated with the Aleph-functions of several variables. *International Journal of Mathematics Trends and Technology (IJMTT)*. 2016 Vol 31 (3), page 142-154.

[2] Chandel R.C.S the products of certain classical polynomials and the generalized Laplacian operator. Ganita 20(1969) p.79-87

[3] Chandel R.C.S and Dwivedi B.N. Generalized Whittaker transforms of hypergeometric functions of several variables. Bull. Inst. Math. Acad . Sinica 8, no4 (1980), p.595-601.

[4] Chandel R.C.S and Dwivedi B.N. Multidimensional Whittaker transforms, Indian. J. Math. 24 (1982), p. 49-53

[5] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[6] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page1-13.

[7] Srivastava H.M.And Panda R. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

Personal adress : 411 Avenue Joseph Raynaud
Le parc Fleuri , Bat B
83140 , Six-Fours les plages
Tel : 06-83-12-49-68
Department : VAR
Country : FRANCE