First Order Stiff Systems On Piecewise Uniform Mesh

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ABSTRACT: In this paper, a piecewise uniform mesh (PUM)is constructed and used, in conjuction with Trapezoidal method to solve the stiff first order system of three Ordinary Differential Equations(ODEs) with complex roots. The capability of the method is proved by the numerical result.

Keywords: Stiff systems, Trapezoidal method, Piecewise uniform mesh.

I. INTRODUCTION

In this paper, we are concerned with the numerical solution of the linear system of first order equations

$$\begin{pmatrix} \frac{du(t)}{dt} \\ \frac{dv(t)}{dt} \\ \frac{dw(t)}{dt} \end{pmatrix} = \begin{pmatrix} f_1(t,u,v,w) \\ f_2(t,u,v,w) \\ f_3(t,u,v,w) \end{pmatrix},$$

$$\text{where} \begin{pmatrix} f_1(t,u,v,w) \\ f_2(t,u,v,w) \\ f_3(t,u,v,w) \end{pmatrix} = A \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix}$$

……………………………………………(1.1)

$$\text{where } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{aconstantmatrixand}$$

$g_1(t), g_2(t), g_3(t)$ are continuous functions of t, where $t \in (0,1]$ with initial values

$$u(0) = \alpha \ , \ v(0) = \beta, \ w(0) = \gamma. \quad \text{………………(1.2)}$$

The system is said to be stiff if the eigenvalues λ_i of the matrix A have negative real parts and for which

$$max_i \,|\, \lambda_i \,| >> min_i \,|\, \lambda_i \,|.$$

a commonly used stiffness index is

$$L = \max |\, Re(\lambda_i) \,|. \quad (1.1)$$

It should be noted that L is not invariant under a simple rescaling of the problem. That raises the distinction between the mathematical problem and the computational problem. The computational problem includes the nature of the error control and the error tolerances; and whether a problem is stiff may depend on this. In particular, rescaling a problem must include a corresponding change of error control if an equivalent problem is to be solved.

Generally, we consider a system to be stiff if the stiffness index is large. More specifically, a system is considered to be stiff on an interval $[t_0, t_f]$ if $L(t_f - t_0) >> 1$. Often the role of the interval is overlooked, though we will see that what might seem a small value of L could contribute to a stiff problem because the interval is long. We have seen examples for which the reverse is true. For instance, a nuclear reactor water hammer modeled by a system of several thousand differential equations had an L that was extremely large. Nevertheless, the model was easily solved using explicit methods because the time interval of interest was a fraction of a millisecond and as a consequence, the problem was not stiff.

For a detailed discussion on stiff nature, application, implicit methods, Trapezoidal method method with UM and Trapezoidal method method with PUM of stiff system of ODEs, one may refer to [1, 2, 3, 5, 7, 8, 10, 11, 12], to name a few.

As in [8], [9] and [10] the focus of this paper is to improve the performance of the Trapezoidal method by applying it in a PUM (Shishkin mesh).

The rest of the paper is organized as follows: In section 2, we present the description of PUM. In section 3, we discuss the order of convergence of the method. In section 4, we establish the efficiency of the method through numerical example. Finally, in section 5, we present some concluding remarks.

The Trapezoidal method is given by

$$\begin{cases} u_{j+1} = u_j + \frac{h}{2}[f_1(t_j, u_j, v_j, w_j) + f_1(t_{j+1}, u_{j+1}, v_{j+1}, w_{j+1})] \\ v_{j+1} = v_j + \frac{h}{2}[f_2(t_j, u_j, v_j, w_j) + f_2(t_{j+1}, u_{j+1}, v_{j+1}, w_{j+1})] \\ w_{j+1} = w_j + \frac{h}{2}[f_3(t_j, u_j, v_j, w_j) + f_3(t_{j+1}, u_{j+1}, v_{j+1}, w_{j+1})], \\ \qquad j = 0 \text{ to } (N-1). \end{cases} \tag{1.3}$$

This is another implicit method which needs a function solve at each step to find u_{j+1}, v_{j+1} and w_{j+1} using a predictor- corrector method and here a corrector is explict Euler method.

$$\begin{cases} u_{j+1} = u_j + hf_1(t_j, u_j, v_j, w_j) \\ v_{j+1} = v_j + hf_2(t_j, u_j, v_j, w_j) \\ w_{j+1} = v_j + hf_3(t_j, u_j, v_j, w_j), \text{where } \; j = 0 \text{to} (N-1). \end{cases} \tag{1.4}$$

2 Discription of PUM

A PUM is constructed on the interval $[0,1]$ as follows:

Choose a transition point σ satisfying $0 < \sigma \le \frac{1}{4}$. Which divides the interval $[0,1]$ into the two subintervals $[0,\sigma]$ and $[\sigma,1]$.

The PUM is constructed by dividing $[0,\sigma]$ into $\frac{N}{4}$ equal mesh elements and $[\sigma,1]$ into $\frac{3N}{4}$ equal mesh elements.

The PUM is used with the following location of the transition point [4]

$$\sigma = \min\{\frac{1}{4}, (\frac{\varepsilon}{\alpha})\, lnN\} \tag{2.1}$$

the parameter ε as

$$\varepsilon < \frac{1}{M}. \qquad (2.2)$$

If A has complex eigenvalues $c_1 \pm ic_2$ then M takes the value of $\sqrt{c_1^2 + c_2^2}$ and α takes the smallest eigenvalue.

$N = 2^m$ *with* $m \geq 7$ (becauseofstiffnesso ftheproblem).

and

$$\begin{cases} t_j = jh_1 \text{where} \quad h_1 = \dfrac{4\sigma}{N}, \quad j = 0(1)\dfrac{N}{4}, \\ t_j = \sigma + (j - \dfrac{N}{4})h_2 \text{where } h_2 = \dfrac{4(1-\sigma)}{3N}, \ j = (\dfrac{N}{4}+1)(1)N. \end{cases} \qquad (2.3)$$

If $\sigma = \frac{1}{4}$ then $h_1 = N^{-1}$ and $h_2 = N^{-1}$.

In such a case the method can be analysed using the standard techniques. We therefore assume that

$$\sigma = \frac{\varepsilon}{\alpha} lnN. \qquad (2.4)$$

3 Order of Convergence

In general, $u(t_{j+1})$ is the exact value and u_{j+1} is the approximate numerical value and the local truncation error at the point $t_{(j+1)}$ in the Trapezoidal method with UM is

$$T_{j+1} = u(t_{j+1}) - u_{j+1} \quad j = 0,1...N-1$$

$$= u(t_{j+1}) - [u(t_j) + \frac{h}{2} f(u(t_{j+1}), t_{j+1}) + \frac{h}{2} f(u(t_j), t_j)]$$

$$= u(t_{j+1}) - [u(t_{j+1}) - hu'(t_{j+1}) + \frac{h^2}{2!}u''(t_{j+1})$$

$$-\frac{h^3}{3!}u'''(t_{j+1})] - \frac{h}{2}u'(t_{j+1}) - \frac{h}{2}u'(t_j)$$

$$= \frac{h}{2}u'(t_{j+1}) - \frac{h^2}{2!}u''(t_{j+1}) + \frac{h^3}{3!}u'''(t_{j+1})$$

$$-\frac{h}{2}[u'(t_{j+1}) - hu''(t_{j+1}) + \frac{h^2}{2!}u'''(t_{j+1})]$$

$$\Rightarrow T_{j+1} = Ch^3 u'''(t_{j+1}).$$

Applying the Trapezoidal method with PUM,

the truncation error for $0 \leq j \leq \frac{N}{4} - 1$ is

$$T_{j+1} = u(t_{j+1}) - u_{j+1} \quad j = 0,1...N-1$$

$$= u(t_{j+1}) - [u(t_j) + \frac{h_1}{2} f(u(t_{j+1}), t_{j+1})$$

$$= u(t_{j+1}) - [u(t_{j+1}) - h_1 u'(t_{j+1}) + \frac{h_1^2}{2!}u''(t_{j+1})$$

$$-\frac{h_1^3}{3!}u'''(t_{j+1})] - \frac{h_1}{2}u'(t_{j+1}) - \frac{h_1}{2}u'(t_j)$$

$$= \frac{h_1}{2}u'(t_{j+1}) - \frac{h_1^2}{2!}u''(t_{j+1}) + \frac{h_1^3}{3!}u'''(t_{j+1})$$

$$-\frac{h_1}{2}[u'(t_{j+1}) - h_1 u''(t_{j+1}) + \frac{h_1^2}{2!}u'''(t_{j+1})]$$

$$\Rightarrow T_{j+1} = Ch_1^3 u'''(t_{j+1}).$$

The truncation error for $\frac{N}{4} \le j \le N-1$ is

$$T_{j+1} = u(t_{j+1}) - u_{j+1} \quad j = 0,1 \ldots N-1$$

$$= u(t_{j+1}) - [u(t_j) + \frac{h_2}{2} f(u(t_{j+1}), t_{j+1})$$

$$+ \frac{h_2}{2} f(u(t_j), t_j)]$$

$$= u(t_{j+1}) - [u(t_{j+1}) - h_2 u'(t_{j+1}) + \frac{h_2^2}{2!} u''(t_{j+1})$$

$$- \frac{h_2^3}{3!} u'''(t_{j+1})] - \frac{h_2}{2} u'(t_{j+1}) - \frac{h_2}{2} u'(t_j)$$

$$= \frac{h_2}{2} u'(t_{j+1}) - \frac{h_2^2}{2!} u''(t_{j+1}) + \frac{h_2^3}{3!} u'''(t_{j+1})$$

$$\Rightarrow T_{j+1} = C h_2^3 u'''(t_{j+1}).$$

Therefore, the truncation error for Trapezoidal method for the component u with PUM is

$$T_{j+1} = \begin{cases} C\, h_1^3\, u'''(t_{j+1}) & for \quad 0 \le j \le \frac{N}{4} - 1 \\ C\, h_2^3\, u'''(t_{j+1}) & for \quad \frac{N}{4} \le j \le N-1 \end{cases}$$

Similarly, the truncation error for v and w can be easily derived.

Wedefine $\| Y \|_s = sup\{| u^{(s)}(t) |, | v^{(s)}(t) |, | w^{(s)}(t) |\}$ $\forall t \in (0,1]$.

Therefore

$$T_{j+1}(h) \le C\, h^3 \, \| Y \|_3, \text{since} h_1 \le h_2 \text{then} h^3 = h_2^{\,3}$$

where $\| Y \|_3 = sup\{| u''' |, | v''' |, | w''' |\}$ forall $t \in (0,1]$.

Hence, by the definition given as in [1], a one step method has order of convergent p, if for any sufficiently smooth solution $y(t)$, there exists constant k and h_0 such that $| T_n(h) | \le k h^{(p+1)}$, $0 < h \le h_0$ (where $T_n(h)$ is the local truncation error), the order of convergence of the Trapezoidal method with the PUM is two. If a linear s-step method is A-stable then it must be an implicit method. Moreover, the order of the method is atmost 2. Therefore Trapezoidal method is A-stable.

4 Numerical examples

In this section, we present an example and it numerical result to illustrate the performance of our method. The numerical results of Trapezoidal method with PUM are compared with UM.

$$\frac{h_2}{2}[u'(t_{j+1}) - h_2 u''(t_{j+1}) + \frac{h_2^2}{2!} u'''(t_{j+1})]$$

The comparison is based in terms of maximum error and average error. The numerical results are recorded interms of the following quantities and tabulated.

As the formula given in [6] for uniform mesh we have,

$h = \frac{(b-a)}{N}$, where b is the end value of t and a is the initial value of t.

(3.1)

The calculation of error (for piecewise uniform mesh and uniform mesh) is given as,

$$error_j = | u(t_j)_{(exact\ solution)} - u_{j(approximate)} |.$$

For maximum error (MAXE) (for piecewise uniform mesh and uniform mesh), we use the formula,

$$MAXE^N \;=\; \max(error_j)$$

The average error for Trapezoidal method with uniform mesh is defined as,

$$AVE = \frac{\sum_{j=1}^{N} error_j}{\frac{(b-a)}{h}},$$

where b is the end value of t and a is the initial value of t.

The average error(AVE) for Trapezoidal method with piecewise uniform mesh is defined as,

$$AVE1 = \frac{\sum_{j=1}^{\frac{N}{4}} (error_j)}{\frac{\sigma}{\frac{h_1}{4}}},$$

$$AVE2 = \frac{\sum_{j=\frac{N}{4}+1}^{N} (error_j)}{\frac{(1-\sigma)}{\frac{3h_2}{4}}} \quad and$$

$$AVE = max\{AVE1, AVE2\}$$

Trapezoidal method works well for a system of three ODEs. To prove this we consider the following example.

Example 4.1

$$u'(t) = -20u - 0.25v - 19.75w, \quad u(0) = 1,$$

$$v'(t) = 20u - 20.25v + 0.25w, \quad v(0) = 0,$$

$$w'(t) = 20u - 19.75v - 0.25w, \quad w(0) = -1$$

$$\text{and} t \in (0,1].$$

Exact solution is:

$$u = [e^{-\frac{1}{2t}} + e^{-20t}(cos(20t) + sin(20t))]/2,$$

$$v = [e^{-\frac{1}{2t}} - e^{-20t}(cos(20t) - sin(20t))]/2,$$

$$w = -[e^{-\frac{1}{2t}} + e^{-20t}(cos(20t) - sin(20t))]/2.$$

The eigenvalues of this problem is

$$-20+20i, -20-20i, -0.5$$

Which are complex and real. So for this problem we consider $\varepsilon = \frac{1}{M}$,

let, the maximum eigen value M be 100 and α as $\frac{1}{2}$.

The numerical results obtained by applying the PUM method to the Examples 4.1 is given in Table 1-3.

5 Conclusion

From the numerical table, we observe that eventhough the number of mesh points are increased in PUM the stability is not lost and the new modified Trapezoidal Method with PUM gives better accuracy than UM for a stiff system of three ordinary differential equations.

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Table 1: *Values of* $MAXE(u), AVE(u)$ *for the solution component* u *for the Example 4.1 for Trapezoidal Method*

	Mesh	MAXE(u)	AVE(u)
	PUM	0.1217e-03	0.1198e-04
	UM	0.2007e-02	0.9455e-04
	PUM	0.4753e-04	0.1921e-05
	UM	0.4800e-03	0.1133e-04
	PUM	0.1890e-04	0.3065e-06
	UM	0.1174e-03	0.1386e-05
	PUM	0.6877e-05	0.4838e-07
	UM	0.2905e-04	0.1715e-06
	PUM	0.2313e-05	0.7524e-08
	UM	0.7223e-05	0.2133e-07
	PUM	0.7347e-06	0.1149e-08
	UM	0.1801e-05	0.2660e-08

	PUM	0.2240e-06	0.1720e-09
	UM	0.4497e-06	0.3321e-09
	PUM	0.6636e-07	0.2527e-10
	UM	0.1124e-06	0.4148e-10
	PUM	0.1926e-07	0.3647e-11
	UM	0.2808e-07	0.5183e-11
	PUM	0.5508e-08	0.5182e-12
	UM	0.7018e-08	0.6478e-12
	PUM	0.1558e-08	0.7269e-13
	UM	0.1754e-08	0.8097e-13

Table 2: *Values of* $MAXE(v), AVE(v)$ *for the solution component* v *for the Example 4.1 for Trapezoidal Method*

N	Mesh	MAXE(v)	AVE(v)
	PUM	0.1208e-03	0.1622e-04
	UM	0.2007e-02	0.8942e-04
	PUM	0.4731e-04	0.2395e-05
	UM	0.4798e-03	0.1074e-04
	PUM	0.1885e-04	0.3476e-06
	UM	0.1174e-03	0.1316e-05
	PUM	0.6864e-05	0.5024e-07
	UM	0.2904e-04	0.1630e-06
	PUM	0.2310e-05	0.7282e-08
	UM	0.7221e-05	0.2027e-07
	PUM	0.7339e-06	0.1061e-08
	UM	0.1801e-05	0.2528e-08
	PUM	0.2238e-06	0.1550e-09
	UM	0.4495e-06	0.3156e-09
	PUM	0.6632e-07	0.2263e-10
		0.1123e-06	0.3943e-10

	UM		
	PUM	0.1925e-07	0.3289e-11
	UM	0.2807e-07	0.4927e-11
	PUM	0.5506e-08	0.4743e-12
	UM	0.7016e-08	0.6158e-12
	PUM	0.1557e-08	0.6776e-13
	UM	0.1754e-08	0.7697e-13

Table 3: *Values of* $MAXE(w), AVE(w)$ *for the solution component* w *for the Example 4.1 for Trapezoidal Method*

N	Mesh	MAXE(w)	AVE(w)
	PUM	0.1208e-03	0.1360e-04
	UM	0.2007e-02	0.9284e-04
	PUM	0.4731e-04	0.2438e-05
	UM	0.4798e-03	0.1109e-04
	PUM	0.1885e-04	0.4060e-06
	UM	0.1174e-03	0.1357e-05
	PUM	0.6864e-05	0.6387e-07
	UM	0.2904e-04	0.1678e-06
	PUM	0.2310e-05	0.9623e-08
	UM	0.7221e-05	0.2086e-07
	PUM	0.7339e-06	0.1403e-08
	UM	0.1801e-05	0.2601e-08
	PUM	0.2238e-06	0.1995e-09
	UM	0.4495e-06	0.3247e-09
	PUM	0.6632e-07	0.2789e-10
	UM	0.1123e-06	0.4056e-10
	PUM	0.1925e-07	0.3854e-11
	UM	0.2807e-07	0.5069e-11
	PUM	0.5506e-08	0.5289e-12
	UM	0.7016e-08	0.6335e-12

	PUM	0.1557e-08	0.7231e-13
	UM	0.1754e-08	0.7918e-13