A note on q –Ruscheweyh type functions

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Abstract — *The objective of the present paper is to define a class $T_q^m(b,\lambda,\alpha)$ using q – Ruscheweyh differential operator . For functions belonging to this class we obtain coefficient estimates, extreme points and many more properties.*

Keywords and phrases — *Univalent functions, q- Ruscheweyh differential operator, q-Komoto operator.*

I. Introduction

q- calculus is a generalization of many subjects , like hyper geometric series, complex analysis and practical physics. In short, q-calculus is quite a popular subject today. It has developed various dialects like quantum calculus, time scales, partitions and continued fractions .

In the present paper , we aim at introducing some new subclasses of functions defined by applying the q- differential operator of order m which are univalent analytic in the open unit disk.

To make this article self contained ,we present below the basic definitions and related details of q-calculus, which are used in sequel.

The q-analogue of gamma function and its integral representation is defined by

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}}$$
$$= (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x,q)_\infty}$$

and $\quad \Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1}E_q(-qt)d_qt$

where $E_q(x)$ is the q-Exponential of x represented by

$$E_q(x) = \sum_{n=0}^{\infty}\frac{q^{\frac{n}{2}}x^n}{(q;q)_n}$$

We observe that

$$as\ q\to 1\ \ E_q(x)\to e(x)\,,\ \Gamma_q(x)\to\Gamma(x)$$

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty}a_nz^n \qquad (1.1)$$

which are analytic and univalent in the open unit disk U.

If a function f is given by (1.1) and g defined by

$$g(z) = z + \sum_{n=2}^{\infty}b_nz^n \qquad (1.2)$$

is in the class $\boldsymbol{A}$, then the convolution (or Hadamard product) of f and g is defined by

$$(\boldsymbol{f}*\boldsymbol{g})(\boldsymbol{z}) = \boldsymbol{z} + \sum_{n=2}^{\infty}\boldsymbol{a_nb_nz^n}\,,\boldsymbol{z}\in\boldsymbol{U} \qquad (\mathbf{1.3})$$

Let T denote the subclass of analytic function in U consisting of functions whose non-zero coefficients from the second onwards are negative .That is an analytic function $f\in T$ if it has a Taylor expansion of the form

$$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{z} - \sum_{n=2}^{\infty}\boldsymbol{a_nz^n}\,,\boldsymbol{a_n}\geq\mathbf{0} \qquad (\mathbf{1.4})$$

which are analytic and univalent in the open disk U.

The class $\boldsymbol{T}$ is also closed under modified Hadamard product , $\boldsymbol{f}*\boldsymbol{g}$ is defined by

$$(f*g)(z) = z - \sum_{n=2}^{\infty}a_nb_nz^n\,,z\in U \qquad (1.5)$$

The q-analogue of derivative of $f(z)$[6] is given by

$$\mathbf{D_qf(z)} = \frac{\mathbf{f(qz)-f(z)}}{\mathbf{z(q-1)}}\,,(\boldsymbol{z}\neq\mathbf{0}\,,\boldsymbol{q}\neq\mathbf{1})$$

Similarly the q-Ruscheweyh differential operator R_q^mf , is defined by

$$\lim_{q\to1}R_q^mf(z) = \frac{z}{(1-z)^{m+1}}*f(z)\,,\quad m>-1$$

Where * denotes convolution .

Now using q-Ruscheweyh differential operator , we define the following subclasses of T.

Definition 1.1 . Let $T_q^m(b,\lambda,\alpha)$ be the subclass of T consisting of functions which satisfy the conditions

$$Re\left\{\frac{(b-2)R_q^mf + 2R_q^{m+1}f}{(b-2\lambda)R_q^mf + 2\lambda R_q^{m+1}f}\right\} > \alpha \qquad (1.6)$$

For some α,$\boldsymbol{\lambda}$,(0≤ $\alpha,\lambda<1$) ,b is a non-zero real number and $z\in U, m>-1$.

For different parametric values of $q\,,m\ and\ b$ we get the classes defined by Mostafa [3] and Shilpa and Latha[4] .

2. Main Results

Theorem 2.1 : A function f defined by (1.4) is in the class $T_q^m(b,\lambda,\alpha)$ if and only if

$$\sum_{n=2}^{\infty} C_q(n,m)a_n\{2(1-\lambda\alpha)([n+m]_q-[1+m]_q) + b(1-\alpha)[1+m]_q\} \le b[1+m]_q(1-\alpha)$$

where $C_q(n,m)=\frac{\Gamma_q(n+m)}{\Gamma_q(1+m)[n-1]!_q}$, $[n]_q=\frac{1-q^{n-1}}{1-q}$, $\alpha,\boldsymbol{\lambda}$, $(0\le\alpha,\lambda<1)$, b is non –zero real number and $z\in U$, $m>-1$.

Proof : Let

$$Re\left\{\frac{(b-2)R_q^m f+2R_q^{m+1}f}{(b-2\lambda)R_q^m f+2\lambda R_q^{m+1}f}\right\}>\alpha$$

which implies

Re

$$\left\{\frac{(b-2)[z-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n]+2\left[z-\sum_{n=2}^{\infty}C_q(n,m)\frac{[n+m]_q}{[1+m]_q}a_nz^n\right]}{(b-2\lambda)[z-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n]+2\lambda\left[z-\sum_{n=2}^{\infty}C_q(n,m)\frac{[n+m]_q}{[1+m]_q}a_nz^n\right]}\right\}>\alpha$$

$$Re\left\{\frac{bz-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n\left[2\frac{[n+m]_q}{[1+m]_q}+b-2\right]}{bz-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n\left[2\lambda\frac{[n+m]_q}{[1+m]_q}+b-2\lambda\right]}\right\}>\alpha$$

Letting $z\to1$, we have

$$b-\sum_{n=2}^{\infty}C_q(n,m)a_n\left[2\frac{[n+m]_q}{[1+m]_q}+b-2\right]>\alpha\left[b-\sum_{n=2}^{\infty}C_q(n,m)a_n\left[2\lambda\frac{[n+m]_q}{[1+m]_q}+b-2\lambda\right]\right]$$

$$\therefore\sum_{n=2}^{\infty}C_q(n,m)a_n\left\{\begin{matrix}2(1-\lambda\alpha)([n+m]_q-[1+m]_q)\\ +b(1-\alpha)[1+m]_q\end{matrix}\right\}\le b[1+m]_q(1-\alpha)$$

Conversely ,suppose $f\in T_q^m(b,\lambda,\alpha)$ satisfies (1.6) Equivalently

$$\left|\left\{\frac{(b-2)R_q^m f+2R_q^{m+1}f}{(b-2\lambda)R_q^m f+2\lambda R_q^{m+1}f}\right\}-1\right|\le1-\alpha$$

$$\left|\left\{\frac{(b-2)[z-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n]+2\left[z-\sum_{n=2}^{\infty}C_q(n,m)\frac{[n+m]_q}{[1+m]_q}a_nz^n\right]}{(b-2\lambda)[z-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n]+2\lambda\left[z-\sum_{n=2}^{\infty}C_q(n,m)\frac{[n+m]_q}{[1+m]_q}a_nz^n\right]}\right\}-1\right|\le\boldsymbol{1-\alpha}$$

$$\left|\left\{\frac{bz-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n\left[2\frac{[n+m]_q}{[1+m]_q}+b-2\right]}{bz-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n\left[2\lambda\frac{[n+m]_q}{[1+m]_q}+b-2\lambda\right]}\right\}-1\right|\le1-\alpha$$

$$\left|\frac{\sum_{n=2}^{\infty}C_q(n,m)a_nz^n2(1-\lambda)([1+m]_q-[n+m]_q)}{bz[1+m]_q-\sum_{n=2}^{\infty}C_q(n,m)a_nz^n[b[1+m]_q-2\lambda([1+m]_q-[n+m]_q)]}\right|\le1-\alpha$$

As $z\to1$

$$\left|\frac{\sum_{n=2}^{\infty}C_q(n,m)a_n2(1-\lambda)([1+m]_q-[n+m]_q)}{b[1+m]_q-\sum_{n=2}^{\infty}C_q(n,m)a_n[b[1+m]_q-2\lambda([1+m]_q-[n+m]_q)]}\right|\le1-\alpha$$

This expression is bounded above by $1-\alpha$, if

$$\sum_{n=2}^{\infty}C_q(n,m)a_n2(1-\lambda)([1+m]_q-[n+m]_q)\le(1-\alpha)\left[b[1+m]_q-\sum_{n=2}^{\infty}C_q(n,m)a_n[b[1+m]_q-2\lambda([1+m]_q-[n+m]_q)]\right]$$

which is true by hypothesis. This completes the assertion of the theorem 2.1 .

As $q\to1$ we get the following result by Shilpa and Laha [4]

Corollary 2.2. A function f defined by (1.4) is in the class $T^m(b,\lambda,\alpha)$ if and only if

$$\sum_{n=2}^{\infty}C(n,m)[2(n-1)(1-\lambda\alpha)+b(1+m)(1-\alpha)]a_n\le b(1+m)(1-\alpha).$$

For $b=2$, $m=1$and $b=1$, $m=1$ in the above theorem we have the following result of Mostafa[3].

Corollary 2.3 . i) A function $f(z)$ defined by (1.4) is in the class $T(\lambda,\alpha)$ if and only if

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n\le1-\alpha$$

ii) A function $f(z)$ defined by (1.4) is in the class $C(\lambda,\alpha)$if and only if

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n\le1-\alpha$$

Corollary 2. 4 . If $f\in T_q^m(b,\lambda,\alpha)$ then

$$|a_n|\le\frac{b[1+m]_q(1-\alpha)}{C_q(n,m)[2(1-\lambda\alpha)([n+m]_q-[1+m]_q)+b(1-\alpha)[1+m]_q]}$$

where $C_q(n,m)=\frac{\Gamma_q(n+m)}{\Gamma_q(1+m)[n-1]_q!}$

Theorem 2.5 . Let $0\le\alpha<1$, $0\le\lambda_1\le\lambda_2<1$, $m>-1$, b is any non-zero real number , then $T_q^m(b,\lambda_2,\alpha)\subset T_q^m(b,\lambda_1,\alpha)$

Proof : From the Theorem 2.1

$$\sum_{n=2}^{\infty}C_q(n\,m)a_n[2(1-\lambda_2\alpha)([n+m]_q-[1+m]_q)+b(1-\alpha)[1+m]_q]$$

$$\le\sum_{n=2}^{\infty}C_q(n\,m)a_n\left[2(1-\lambda_1\alpha)\begin{pmatrix}[n+m]_q\\-[1+m]_q\end{pmatrix}+b(1-\alpha)[1+m]_q\right]$$

$$\leq b[1+m]_q(1-\alpha)$$

For $f(z) \in T_q^m(b,\lambda_2,\alpha)$,

Hence $f(z) \in T_q^m(b,\lambda,\alpha)$.

Theorem 2.6. Let $f(z) \in T_q^m(b,\lambda,\alpha)$ define $f_1(z) = z$ and

$$f_n(z) = z - \frac{b[1+m]_q(1-\alpha)}{C_q(n,m)[2(1-\lambda\alpha)([n+m]_q-[1+m]_q)+b(1-\alpha)[1+m]_q]} z^n$$

Where $n = 2,3, \ . \ . \ . \ , \ z \in U$,

$$C_q(n,m) = \frac{\Gamma_q(n+m)}{\Gamma_q(1+m)\ [n-1]_q!}$$

α, $(0 \leq \alpha < 1)$, $\lambda\ (0 \leq \lambda < 1)$, b is non –zero real number and $z \in U, m > -1$

$f \in T_q^m(b,\lambda,\alpha)$ if and only if f can be expressed as

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \text{ , where } \mu_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \mu_n = 1$$

Proof : If

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \text{ , with } \mu_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \mu_n = 1$$

Then

$$\sum_{n=2}^{\infty} \frac{C_q(n,m)[2(1-\lambda\alpha)[[n+m]_q-[1+m]_q]+b(1-\alpha)[1+m]_q]}{C_q(n,m)[2(1-\lambda\alpha)[[n+m]_q-[1+m]_q]+b(1-\alpha)[1+m]_q]}$$

$$= \sum_{n=2}^{\infty} \mu_n b[1+m]_q(1-\alpha)$$

$$= (1-\mu_1)b[1+m]_q(1-\alpha)$$
$$\leq b[1+m]_q(1-\alpha)$$

Hence $f \in T_q^m(b,\lambda,\alpha)$

Conversely ,

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ \in \ T_q^m(b,\lambda,\alpha)$

Define

$$\mu_n = \frac{C_q(n,m)[2(1-\lambda\alpha)([n+m]_q-[1+m]_q)+b\,[1+m]_q(1-\alpha)]|a_n|}{b[1+m]_q(1-\alpha)}$$

$m = 2,3, \ . \ . \ .$

And define $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$,

from Theorem 2.1 , $\sum_{n=2}^{\infty} \mu_n \leq 1$ and so $\mu_1 \geq 0$

Since $\mu_n f_n(z) = \mu_n f_1 - a_n z^n$

$$\sum_{n=1}^{\infty} \mu_n f_n(z) = z - \sum_{n=2}^{\infty} a_n z^n = f(z).$$

Theorem 2.7. The class $T_q^m(b,\lambda,\alpha)$ is closed under convex linear combination .

Proof : Let $f, g \in T_q^m(b,\lambda,\alpha)$, and let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ , \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

For η such that $0 \leq \eta \leq 1$, it is enough to show that the function defined by

$$h(z) = (1-\eta)f(z) + \eta g(z), z \in U$$

belongs to $T_q^m(b,\lambda,\alpha)$

$$\text{now } h(z) = z - \sum_{n=2}^{\infty} [(1-\eta)a_n + \eta b_n] z^n$$

Applying Theorem 2.1 to $f, g \in T_q^m(b,\lambda,\alpha)$ we have

$$\sum_{n=2}^{\infty} C_q(n,m) \begin{Bmatrix} 2(1-\lambda\alpha)([n+m]_q-[1+m]_q) \\ +b[1+m]_q(1-\alpha) \end{Bmatrix}$$

$$[(1-\eta)a_n + \eta b_n]$$

$$= (1-\eta) \sum_{n=2}^{\infty} C_q(n,m) \begin{bmatrix} 2(1-\lambda\alpha)([n+m]_q-[1+m]_q) \\ +b[1+m]_q(1-\alpha) \end{bmatrix} a_n$$

$$+\eta \left[\sum_{n=2}^{\infty} C_q(n,m) \begin{bmatrix} 2(1-\lambda\alpha)([n+m]_q-[1+m]_q) \\ +b[1+m]_q(1-\alpha) \end{bmatrix} \right] b_n$$

$$\leq (1-\eta)b[1+m]_q(1-\alpha) + \eta b[1+m]_q(1-\alpha)$$
$$= b[1+m]_q(1-\alpha).$$

This implies that $h \in T_q^m(b,\lambda,\alpha)$.

Corollary 2.8 . *If* $f_1(z), f_2(z) \in T_q^m(b,\lambda,\alpha)$ then the function defined by

$$g(z) = \frac{1}{2}[f_1(z) + f_2(z)] \in T_q^m(b,\lambda,\alpha)$$

Theorem 2.9 . Let for $j = 1,2, . \ . \ . \ n$.

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \in T_q^m(b,\lambda_j,\alpha) \ , 0 < \lambda_j < 1$$

such that $\sum_{j=1}^{n} \lambda_j = 1$, then the function F(z) is defined by

$$F(z) = \sum_{j=1}^{n} \lambda_j f_j(z) \ \in T_q^m(b,\lambda,\alpha).$$

Proof : For each $j \in \{1,2,\ldots n\}$ we obtain

$$\sum_{n=2}^{\infty} C_q(n,m) \begin{bmatrix} 2(1-\lambda\alpha)([n+m]_q-[1+m]_q) \\ +b[1+m]_q(1-\alpha) \end{bmatrix} |a_n|$$

$$< b[1+m]_q(1-\alpha)$$

$$F(z) = \sum_{j=1}^{n} \lambda_j \left[z - \sum_{n=2}^{\infty} a_{n,j} z^n \right]$$

$$= z - \sum_{n=2}^{\infty}\left[\sum_{j=1}^{n}\lambda_j a_{n,j}\right]z^n$$

$$\sum_{n=2}^{\infty} C_q(n,m)\big[2(1-\lambda\alpha)([n+m]_q - [1+m]_q) + b[1+m]_q(1-\alpha)\big]\left[\sum_{j=1}^{n}\lambda_j a_{n,j}\right]$$

$$= \sum_{j=1}^{n}\lambda_j\left[\sum_{n=2}^{\infty} C_q(n,m)\left[2(1-\lambda\alpha)\begin{pmatrix}[n+m]_q\\-[1+m]_q\end{pmatrix} + b[1+m]_q(1-\alpha)\right]\right]a_{n,j}$$

$< b[1+m]_q(1-\alpha)$.

Therefore $F(z) \in T_q^m(b,\lambda,\alpha)$.

Definition 1.2 . The q-Komoto operator of f is defined by

$K_q(z)$

$$= \int_{qE_q\left(\frac{-q}{(c+1)(1-q)}\right)}^{q} \frac{(c+1)^{\Upsilon}}{q^{c+1+\Upsilon}\Gamma_q(\Upsilon)}\, t^c \left[\mathrm{Log}_q \frac{q}{t}\right]^{\Upsilon-1} \frac{f(tz)}{t}\, d_q t$$

$c > -1$, $\Upsilon \geq 0$.

Theorem 2.11 . Let $f(z) \in T_q^m(b,\lambda,\alpha)$ then $K_q(z) \in T_q^m(b,\lambda,\alpha)$.

Proof : we have

$$\int_{qE_q\left(\frac{-q}{(c+1)(1-q)}\right)}^{q} t^c\left(\mathrm{Log}_q \frac{q}{t}\right)^{\Upsilon-1} d_q t = \frac{q^{c+1+\Upsilon}}{(c+1)^{\Upsilon}}\,\Gamma_q(\Upsilon)$$

$$\int_{qE_q\left(\frac{-q}{(c+1)(1-q)}\right)}^{q} t^{n+c-1}\left(Log_q \frac{q}{t}\right)^{\Upsilon-1} d_q t = \frac{q^{c+n+\Upsilon}}{(c+n)^{\Upsilon}}\,\Gamma_q(\Upsilon)$$

$K_q(z)$

$$= \frac{(c+1)^{\Upsilon}}{q^{c+1+\Upsilon}\Gamma_q(\Upsilon)}\left[\int_{qE_q\left(\frac{-q}{(c+1)(1-q)}\right)}^{q} t^c\left(Log_q \frac{q}{t}\right)^{\Upsilon-1} z\, d_q t - \sum_{n=2}^{\infty} z^n \int_{qE_q\left(\frac{-q}{(c+1)(1-q)}\right)}^{q} a_n\, t^{n+c-1}\left(Log_q \frac{q}{t}\right)^{\Upsilon-1} d_q t\right]$$

$$= z - \sum_{n=2}^{\infty} q^{n-1}\left(\frac{c+1}{c+n}\right)^{\Upsilon} a_n z^n$$

Since $f \in T_q^m(b,\lambda,\alpha)$ and $0 < q < 1$, $\left(\frac{c+1}{c+n}\right)^{\Upsilon} < 1$, we have

$$\sum_{n=2}^{\infty} C_q(n,m)\begin{bmatrix}2(1-\lambda\alpha)([n+m]_q - [1+m]_q) +\\ b[1+m]_q(1-\alpha)\end{bmatrix} q^{n-1}\left(\frac{c+1}{c+n}\right)^{\Upsilon} a_n$$

$< b[1+m]_q(1-\alpha)$

Therefore $K_q(z) \in T_q^m(b,\lambda,\alpha)$.

Theorem 2.12 . Let $f \in T_q^m(b,\lambda,\alpha)$ then for every $0 \leq \beta < 1$, the function

$$H_\beta(z) = (1-\beta)f(z) + \beta\int_0^z \frac{f(t)}{t}\, d_q t$$

then $H_\beta(z) \in T_q^m(b,\lambda,\alpha)$.

Proof : we have

$$H_\beta(z) = z - \sum_{n=2}^{\infty}\left(1 + \frac{\beta}{n} - \beta\right)a_n z^n$$

Since $\left(1 + \frac{\beta}{n} - \beta\right) < 1$, $n \geq 2$, so by Theorem 2.1

$$\sum_{n=2}^{\infty}\left(1 + \frac{\beta}{n} - \beta\right)C_q(n,m)\left[2(1-\lambda\alpha)\begin{pmatrix}[n+m]_q\\-[1+m]_q\end{pmatrix} + b[1+m]_q(1-\alpha)\right]a_n$$

$$< \sum_{n=2}^{\infty} C_q(n,m)\begin{bmatrix}2(1-\lambda\alpha)([n+m]_q - [1+m]_q)\\ +b[1+m]_q(1-\alpha)\end{bmatrix}a_n$$

$< b[1+m]_q(1-\alpha)$

Hence $H_\beta(z) \in T_q^m(b,\lambda,\alpha)$

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