A work on Representation Theorem's and Algebraic Applications

sahar jaafar mahmood abumalah[1]

Assistant lecturer, Mathematics and University of Al_Qadisiya

Iraq

Abstract: *In mathematics a representation theorem is a theorem which states that each abstract structure with certain properties is isomorphic to a concrete structure. There are several examples of theorems of representation in various fields of mathematics: . In algebra, the theorem of Cayley states that each group is isomorphic to a group transformed a whole. The representation theory studies the properties of abstract groups through their representation as transformations of vector spaces. Additionally, also in algebra, Stone representation theorem for Boolean algebras states that every Boolean algebra is isomorphic to a field of sets. A variant of this theorem channeled reticles requires each distributive lattice is isomorphic to a reticle sub-lattice power set of a set. . In category theory, the Yoneda lemma explains how arbitrary functors in the category of sets can be seen as hom functions. . In set theory, the Mostowski collapsing theorem states that every well-founded extensional structure is isomorphic to a transitive set with the relation of belonging (∈). In functional analysis, the Riesz representation theorem is currently a list of many theorems. One identifies the dual space C_0 (X) with the set of regular measurements in X.in geometry, the Whitney embedding theorems embed any abstract manifold in some Euclidean space. The Nash embedding theorem embeds an abstract Riemannian manifold isometrically in a Euclidean space. But the aim of this work is the algebraic applications, specially the* representation theory.

I. Introduction

Representation theory is remarkable for its abundance of branches and diversity of approaches. Although all theories have in common the basic concepts presented above, they are significantly different in their details. These differences are at least three types: .The performances depend on the nature of algebraic objects represented and have different characteristics according to family groups, associative algebras or Lie algebra that consideration. .They also depend on the type of vector spaces considered. The most important distinction between the representations of finite degree and those of infinite degree is found. We may impose additional structures on the space (Hilbert, Banach, etc. in the infinite case, algebraic structures in the finite case). .Finally, they depend on the type of the base member K. A highly studied case is that of the fields' complex. Other important cases are the field of real, finite fields and the fields of p-adic numbers. Additional difficulties arise when K is a positive characteristic or is not algebraically closed. We will discuss about two important algebra representation theorems.

II. Theorem of Cayley

Cayley's theorem is a basic result establishing that any group is realized as a group of permutations, that is to say as a subgroup of a symmetric group:

Any group G is isomorphic to a subgroup of a symmetric group S (G) of permutations of G. In particular, if G is a finite group of order n is isomorphic to a subgroup of S_n.

2.1 Proof

Let G be a group and g an element of this group. We define the t_g mapping of G into G as the left translation:

$\forall x \in G \; t_g(x) = gx$.

The associativity of the group's law is equivalent to:

$\forall g, h \in G \; t_{gh} = t_g \circ t_h$. (∗)

It is deduced in particular that t_g is a permutation (inverse bijection of t_g^{-1}), thereby defining a mapping t of G in S (G) by:

$\forall g \in G \; t(g) = t_g$.

By (*) t is a group morphism.

Its kerf is the trivial group {e} (where e denotes the neutral element of G) because if an element g of G is such that t_g is the application identity then g = tan (e) = e.

According to the first isomorphism theorem, t therefore realizes an isomorphism between G and the subgroup Im (t) S (G).

Remarks

If G is of order n, the group Sn in which it is immersed is of order n!

The theorem is reformulated by saying that every group acts faithfully on itself. The action that we built is actually even simply transitive.

2.2 Applications

This theorem is used in the theory of group representations. Let G be a group and $(e_g)_{g \in G}$ a basis

of a vector space of dimension | G |. The Cayley theorem states that G is isomorphic to a group of permutations of the basic elements. Each permutation can be extended to an endomorphism of E that here, by construction, is an automorphism of E. This defines a representation of the group: its regular representation.

It is also involved in a demonstration of the first Sylow theorem.

III. Stone's representation theorem for Boolean algebras

Stone's representation theorem for Boolean algebras states that every Boolean algebra is isomorphic to a field of sets. The theorem is fundamental to the deeper comprehension of Boolean algebra that came into existence in the first half of the 20th century. First of all, Marshall H. Stone proved the theorem.

The theorem was first proved by Marshall H. Stone (1936), and thus named in his honor. Stone was led to it by his study of the spectral theory of operators on a Hilbert space.

3.1 Stone spaces

Each Boolean algebra B has an associated topological space, denoted here S(B). This is known as its Stone space. In S(B), the points are known as ultra filters that are associated with B. In other words homomorphisms from B and it are too related to the two-element Boolean algebra. On the closed basis the topology is generated on S(B) which is expressed as

$\{x \in S(B) \backslash b \in x\}$

where b is an element of B.

For every Boolean algebra B, S(B) is a compact totally disconnected Hausdorff space; such spaces are called Stone spaces (also profinite spaces). Conversely, given any topological space X, the collection of subsets of X that are clopen (both closed and open) is a Boolean algebra.

3.2 Representation theorem (Stone)

A simple version of Stone's representation theorem states that every Boolean algebra which is denoted as B is said to be isomorphic with respect to the algebra of clopen subsets associated with Stone space S(B). It is observed that an isomorphism is able to send an element b ∈ B to different ultra filters that can have the b. This kind of set is known as clopen set as per the topology S(B) another fact is that B belongs to Boolean algebra.

When the theorem is restated with the help of the language of category theory, it is said that the theorem reflects a duality between Boolean algebra's category and Stone spaces category. The meaning of the duality refers to the fact that each homomorphism from A to B indicates there is natural way which leads to continuous function form S(B) and to S(A) where A and B are considered Boolean algebra. Stated differently between the two categories equivalence is achieved by using a contravariant functor. This can be considered as an example determined early for nontrivial duality of categories.

The theorem considered is considered a special case with respect to Stone duality. It is also said to be a more general framework with respect to dualities between partially ordered sets and topological spaces. The proof for the theorem needs either weakened form of it or the axiom of choice. It is observed that this theorem is equal to the prime ideal theorem. According to that it is a weakened choice principle. It indicates that each and every Boolean algebra can be considered as a prime ideal.

IV. Representation Theory

4.1 Definitions and Concepts

Let V be a vector space over a field K[3] . For example, assume that V is $\mathbb{R}^n$ or $\mathbb{C}^n$, ordinary space of dimension n column vectors on the body ℝ real or that, ℂ, complex. In this case, the idea of representation theory is to make abstract algebra concretely, using n × n matrices of real or complex numbers. We can do it for three main types of algebraic objects: groups, associative algebra and algebra of Lie.

The subset of invertible n × n matrices form a group for multiplication and group representations theory analysis describing a group by "representative" - its elements in terms of singular matrices.

Addition and multiplication are the set of all n × n matrices associative algebra, which gives rise to the theory of algebraic representations associative.

If we replace the MN product of two matrices by their commutate MN − NM, then the n × n matrices do not form an associative algebra but a Lie algebra, and we study the representations of Lie algebra.

This generalizes to any field K and any vector space V over K, by replacing the matrices by linear endomorphisms and the matrix produced by the composition: the automorphisms of V form the group GL (V) and the endomorphisms of V form the associative algebra End (V), which corresponds to the Lie algebra gl (V).

Definitions

There are two ways to explain what a representation

The first uses the concept of Action, generalizing how the matrices act by product on the column vectors. A representation of a group G or an algebra A (associative or Lie) on a vector space V is applied

$\Phi: G \times V \to V$ or $\Phi: A \times V \to V$

with two properties. First, for every x in G or A, application

$\varphi(x): V \to V$

$v \mapsto \Phi(x, v)$

is K -linear. Second, by introducing the notation $g \cdot v$ for Φ (g, v), we have, for all g_1, g_2 in G and all v in V:

(1) $e. v = v$

(2) $g_1. (g_2. v) = (g_1 g_2). v$

where e is the neutral element of G and g_1 g_2 is the product in G. The condition for an associative algebra A is similar, except that A cannot have element unit, in which case equation (1) is omitted. Equation (2) is an abstract expression of the associativity of the matrix product. It is not verified by the matrix switches and switches for these, there is no neutral element, so that for the Lie algebra, the only condition is that for all x_1, x_2 in A and all v in V:

(2′) $x_1. (x_2. v) - x_2. (x_1. v) = [x_1, x_2]. v$

where $[x_1, x_2]$ is the Lie bracket, which generalizes the switch MN − NM of two matrices.

The second way, more concise and more abstract, focuses on the application φ that at any x of G or A combination φ (x): V → V, which must verify for all x 1, x 2 in Group Gor associative algebra A:

(2) $\varphi(x_1, x_2) = \varphi(x_1) \circ \varphi(x_2)$.

A representation of a group G on a vector space V is a group morphism φ: G → GL (V).

A representation of an associative algebra A on V is an algebra morphism φ: A → End (V).A representation of a Lie algebra was on V is a Lie algebra morphism φ: → (V).

4.2 Terminology

Representation X (V, φ), or simply V if the morphism φ is clear from the context. The vector space V is called the space of representation (V, φ) and the size n of V is called the degree of (V, φ).

If this degree n is finite, the selection of a basis of V identifies V to K^n and found a representation by matrices with entries in K.

The representation (V, φ) is called faithful if the morphism φ is injective. For an associative algebra, the concept equivalent to that of faithful Module. For representation of a group G on a K vector -space, if the associated representation of the algebra K [G] is true then the representation of G is too, but the converse is false, as shown by the example of the regular representation of the symmetric group S_4.

4.3 Morphisms

If (V, φ) and (W, ψ) are two representations of a group G,

called intertwining operator, or morphism of representations from the first to the second all linear α: V We equivariantly, c ′ is to say such that for each g in G and all v in V,

$\alpha(g. v) = g. \alpha(v)$

which, in terms of morphisms φ: G → GL (V) and ψ: G → GL (W), is written:

$\forall g \in G, \varphi \circ \varphi(g) = \psi(g) \circ \alpha$

We define the same morphisms of representations of an associative algebra or Lie.

If α is invertible, they say it is an isomorphism of representations and that the two representations are isomorphic. They are then, from a practical point of view, "identical": they provide the same information about the group or algebra they represent. Therefore representations theory seeks to classify the representations "isomorphism".

Morphisms of a representation to itself are called its endomorphisms. They form an associative algebra over the base member K.

4.4 Set-representations

A representation "of sets," or representation "by permutations" of a group G (as opposed to the representations referred to above, so-called "linear") is an action of G on a set X, that is to say the data an application ρ, of G in all X^X all applications X in X such that for all g 1, g 2 in G and x in X:

$\rho(1)(x) = x$ and $\rho(g_1g_2)(x) = \rho(g_1)(\rho(g_2)(x))$,
where 1 means the neutral element of the group G.

These conditions, together with the definition of a group, cause that ρ (g) (for any g in G) are bijections, so that an equivalent definition of a representation of G by permutation is given by a group morphism G in the symmetric group S_X of X.

V. Conclusion

Representation theory is a powerful tool, because it reduces the problems of abstract algebra to linear algebra problems, an area that is well included. Moreover, when allowing the vector space on which a group (for example) is shown to be an infinite dimensional space, for instance a Hilbert space can be applied to the theory of groups of analysis methods. Representation theory is as important in physics because it helps to describe, for example, how the symmetry group of a system affects the solutions of the equations described.
A striking feature of the representation theory is its ubiquity in mathematics. This has two aspects. First, the applications of this theory are varieties: in addition to its impact in algebra, it enlightens and widely generalized Fourier analysis via harmonique analysis, it is deeply linked to the geometry via the invariant theory and Erlangen to program and has a profound impact in number theory via automorphic forms and Langlands program. The second aspect of the ubiquity of representation theory is the variety of ways to approach it. The same objects can be studied using the methods of Algebraic Geometry, Theory modules, analytical number theory, differential geometry, operator theory (in) and topologies.
The success of the representation theory has led to numerous generalizations. One of the general's category. Algebraic objects that theory applies can be seen as special cases of categories, and representations as functors, such a category in the vector spaces. This description indicates two obvious generalizations: first, algebraic objects can be replaced by more general categories and secondly, the finish category of vector spaces can be replaced by other categories that we control well.

References

[1] J. L. Alperin, Local Representation Theory: Modular Representations as an Introduction to the Local Representation Theory of Finite Groups, , 1986

[2] Armand Borel, Essays in the History of Lie Groups and Algebraic Groups, AMS, 2001

[3] Armand Borel et W. Casselman, Automorphic Forms, Representations, and L-functions, AMS, 197

[4] Charles W. Curtis et Irving Reiner, Representation Theory of Finite Groups and Associative Algebras, John Wiley & Sons, 1962

[5] William Fulton et Joe Harris, Representation Theory : A First Course

[6] Stephen Gelbart, « An Elementary Introduction to the Langlands Program », Bull. Amer. Math. Soc., vol. 10, n° 2, 1984, p. 177-219

[7] Roe Goodman et Nolan R. Wallach, Representations and Invariants of the Classical Groups, CUP, 1998

[8] James Gordon et Martin Liebeck, Representations and Characters of Groups, CUP, 1993

[9] James E. Humphreys , Introduction to Lie Algebras and Representation, Theory, Birkhäuser, 1972

[10] James E. Humphreys, Linear Algebraic Groups, Springer, coll. « GTM » (n° 21), 1975

[11] Jens Carsten Jantzen , Representations of Algebraic Groups, AMS, 2003

[12] Anthony W. Knapp **(de)**, Representation Theory of Semisimple Groups : An Overview Based on Examples,, 2001

[13] T. Y. Lam, « Representations of Finite Groups: A Hundred Years », Notices Amer. Math. Soc., vol. 45, n° 3 et 4, 1998, 361-372 (Part I), 465-474 (Part II)

[14] Peter J. Olver, Classical Invariant Theory, CUP, 1999

[15] Jean-Pierre Serre, Linear Representations of Finite Groups

[16] R. W. Sharpe, Differential Geometry : Cartan's Generalization of Klein's Erlangen Program, Springer, coll. « GTM » (n° 166), 1997

[17] Daniel Simson, Andrzej Skowronski et Ibrahim Assem, Elements of the Representation Theory of Associative Algebras, CUP, 2007

[18] S. Sternberg, Group Theory and Physics, CUP, 1994

[19] Introduction to representation theory Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina January 10, 2011

[20] Arthur Cayley, "On the theory of groups, as DEPENDING on the symbolic equation θn = 1", Philos. Mag., Vol. 7, No. 4, 1854, p. 40-47.

[21] William Burnside, Theory of Groups of Finite Order, Cambridge, 2004 (1st ed. 1911) , p. 22.

[22] Camille Jordan, Treaty substitutions and algebraic equations, Paris, Gauther-Villars, 1870, p. 60-61.