Some Statistical Properties of the Solutions of a System of two dimensional Integral Equations contains Beta distribution

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Abstract

In this paper, we find the two solutions of a system of two dimensional integral equations contains respectively two beta distributions with different values of the two parameters in two cases and equal in the third that is to introduce two corresponding probability density functions. The variance of each resulting probability density function corresponding to the supposing three cases are derived that is to indicate which of the probability density function gave the maximum variance. Furthermore, the correlation coefficient between any pair of the probability density function is determined.

Key words: *Beta distribution, Adomian decomposition method, System of two dimensional integral equations.*

1- INTRODUCTION

Many researchers interested as a final goal either by studying the existence and uniqueness of the solution of many types of one or more dimensional integral equations [1,2] or to find by using different methods of the modified quadrature of the numerical solutions of this kind of equations on some definite closed interval to study a comparison between the numerical solutions and their exact solutions [3]. While [4] use the Adomian decomposition method to compare this method with the classical successive method for solving system of linear Fredholm integral equations and [5] studied the comparison between Newton's method and Adomian decomposition method for solving special Fredholm integral equations.

For combining the integral equations as an important branch of mathematics with some statistical distributions, our goal in this study is not only interesting in the solutions of the supposing a system of two-dimensional integral equations contains two beta distributions with different values of the two parameters but we concentrate ourselves in the derivation of some statistical properties of the solutions like probability density functions and their variances when the two solutions of the system are found by the Adomian decomposition method [6].

2- PRELIMINARIES

The Beta distribution is a continuous probability function with the following marginal probability density function

$$g(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\ ,0<x<1,\alpha,\beta>0 \qquad \ldots(2.1)$$

where α, β are two shape parameters.

The Beta distribution is characterized by the following properties, [2]

(i) $M_x(t) = 1 + \sum_{n=1}^{\infty}\left[\prod_{m=0}^{n-1}\frac{\alpha+m}{\alpha+\beta+m}\right]\frac{t^n}{n!}$

(ii) $E(x) = \frac{\alpha}{\alpha+\beta}$

(iii) $\mathrm{var}(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Now, we consider the following two dimensional integral equations:

$$f_j(x) = g_i(x) + \int_0^1 \sum_{j=1}^{2} k_{ij}(x,s) f_j(s)\,ds \qquad \ldots(2.2)$$

where,

- ❖ $k_{ij}(x,s)$ are known kernel defined by $0 < x < 1$, $0 < s < 1$, $s \in S$, S is a compact metric space and having respectively the supposing formulas:

$$\left.\begin{array}{l} k_{1j}(x,s) = se^{-x} \\ \qquad\qquad\qquad j = 1,2 \\ k_{2j}(x,s) = se^{-x^2} \end{array}\right\} \qquad \ldots(2.3)$$

- ❖ $f_j(s)$, $j = 1,2$ are scalar functions defined for $0 < s < 1$.

By substituting (2.1), (2.3) into (2.2), we get:

$$\left.\begin{array}{l} f_1(x) = \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} x^{\alpha_1-1}(1-x)^{\beta_1-1} + \int_0^1 se^{-x}\; f_1(s)+f_2(s)\; ds \\ f_2(x) = \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} x^{\alpha_2-1}(1-x)^{\beta_2-1} + \int_0^1 se^{-x^2}\; f_1(s)+f_2(s)\; ds \end{array}\right\} \ldots(2.4)$$

Remark (2.1):

In this paper three cases as a sample from nine for the parameters (α,β) should be considered:

1. $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$.
2. $\alpha_1 > \beta_1 : \alpha_1 = 2, \beta_1 = 1; \alpha_2 < \beta_2 : \alpha_2 = 1, \beta_2 = 2$.
3. $\alpha_1 < \beta_1 : \alpha_1 = 1, \beta_1 = 2; \alpha_2 > \beta_2 : \alpha_2 = 2, \beta_2 = 1$.

Now, to find the two solutions of the system (2.4), Adomian decomposition method should be used which briefly depends on the following steps, [5]:

$$\left.\begin{array}{l} g_{10}(x) = \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} x^{\alpha_1-1}(1-x)^{\beta_1-1} \\ g_{20}(x) = \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} x^{\alpha_2-1}(1-x)^{\beta_2-1} \end{array}\right\} \ldots(2.5)$$

$$\left.\begin{array}{l} g_{10}(s) = \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} s^{\alpha_1-1}(1-s)^{\beta_1-1} \\ g_{20}(s) = \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} s^{\alpha_2-1}(1-s)^{\beta_2-1} \end{array}\right\} \ldots(2.6)$$

And

$$\left.\begin{array}{l} f_{1,m+1}(x) = \int_0^1 se^{-x}\; f_{1m}(s)+f_{2m}(s)\; ds \\ f_{2,m+1}(x) = \int_0^1 se^{-x^2}\; f_{1m}(s)+f_{2m}(s)\; ds \end{array}\right\} \ldots(2.7)$$

That is to get the following two solutions:

$$\left.\begin{array}{l} f_1(x) = g_{10}(x) + \sum_{n=1}^{\infty} f_{1n}(x) \\ f_2(x) = g_{20}(x) + \sum_{n=1}^{\infty} f_{2n}(x) \end{array}\right\} \ldots(2.8)$$

So, for the first iteration (m = 0), by (2.6) and (2.7):

$$f_{11}(x) = \int_0^1 se^{-x}\left[\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} s^{\alpha_1-1}(1-s)^{\beta_1-1} + \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} s^{\alpha_2-1}(1-s)^{\beta_2-1}\right]ds$$

$$f_{21}(x) = \int_0^1 se^{-x^2}\left[\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} s^{\alpha_1-1}(1-s)^{\beta_1-1} + \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} s^{\alpha_2-1}(1-s)^{\beta_2-1}\right]ds$$

or

$$\left.\begin{array}{l} f_{11}(x) = \left(\frac{\alpha_1}{\alpha_1+\beta_1} + \frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x} \\ f_{21}(x) = \left(\frac{\alpha_1}{\alpha_1+\beta_1} + \frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x^2} \end{array}\right\} \ldots(2.9)$$

and for the second iteration (m = 1), by (2.7) and (2.9),

$$f_{12}(x) = \int_0^1 se^{-x}\left[\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s} + \left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s^2}\right]ds$$

$$f_{22}(x) = \int_0^1 se^{-x^2}\left[\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s} + \left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s^2}\right]ds$$

or

$$\left.\begin{aligned} f_{12}(x) &= 0.5803\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x} \\ f_{22}(x) &= 0.5803\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x^2} \end{aligned}\right\} \quad \ldots(2.10)$$

and for the third iteration (m = 2), by (2.7) and (2.10),

$$f_{13}(x) = \int_0^1 se^{-x}\left[0.5803\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s} + 0.5803\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s^2}\right]ds$$

$$f_{23}(x) = \int_0^1 se^{-x^2}\left[0.5803\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s} + 0.5803\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-s^2}\right]ds$$

or

$$\left.\begin{aligned} f_{13}(x) &= (0.5803)^2\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x} \\ f_{23}(x) &= (0.5803)^2\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x^2} \end{aligned}\right\} \ldots(2.11)$$

By repeating iterations for m = 3, 4, …, we get:

$$\left.\begin{aligned} \sum_{n=1}^{\infty} f_{1n}(x) &= \left[\sum_{k=0}^{\infty} (0.5803)^k\right]\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x} \\ \sum_{n=1}^{\infty} f_{2n}(x) &= \left[\sum_{k=0}^{\infty} (0.5803)^k\right]\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x^2} \end{aligned}\right\}$$

where, the geometrical series, $\sum_{k=0}^{\infty} (0.5803)^k = 2.38265$.

So,

$$\left.\begin{aligned} \sum_{n=1}^{\infty} f_{1n}(x) &= 2.38265\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x} \\ \sum_{n=1}^{\infty} f_{2n}(x) &= 2.38265\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x^2} \end{aligned}\right\} \ldots(2.12)$$

Finally, by substituting (2.5) and (2.12) into (2.8) which represent the two solutions of (2.4), we get:

$$\left.\begin{aligned} f_1(x) &= \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)}x^{\alpha_1-1}(1-x)^{\beta_1-1} + 2.38265\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x} \\ f_2(x) &= \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)}x^{\alpha_2-1}(1-x)^{\beta_2-1} + 2.38265\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x^2} \end{aligned}\right\} \ldots(2.13)$$

$0 < x < 1$, α_i , $\beta_I > 0$, i = 1, 2.

3- STATISTICAL PROPERTIES OF THE SOLUTIONS:

In order to introduce the statistical properties of the two solutions (2.13), it must be that each of them is a p.d.f. of a continuous r.v. x. So, we multiply them respectively by A and B and equate their integrals by one that is to find A and B which make each solution is a p.d.f. (i.e.), we write:

$$\int_0^1 Af_1(x)\,dx = 1 \ , \quad \int_0^1 Bf_2(x)\,dx = 1$$

$$A\left[\int_0^1 \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} x^{\alpha_1-1}(1-x)^{\beta_1-1}dx + \int_0^1 2.38265\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x}dx\right]=1$$

$$A+(2.38256)(0.63212)\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)A=1$$

So,

$$A=\frac{1}{1+1.50606\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)}$$

while, the second integral gives the following value of B

$$B=\frac{1}{1+1.77934\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)}$$

Hence, the two solutions (2.13) are probability density functions of the random variable X that are when:

$$\Phi_1(x)=\frac{\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)}x^{\alpha_1-1}(1-x)^{\beta_1-1}+2.38265\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x}}{1+1.50606\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)} \quad \ldots(2.13a)$$

$$\Phi_2(x)=\frac{\frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)}x^{\alpha_2-1}(1-x)^{\beta_2-1}+2.38265\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)e^{-x^2}}{1+1.77934\left(\frac{\alpha_1}{\alpha_1+\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right)} \quad \ldots(2.13b)$$

where, $0 < x < 1$, α_i, $\beta_i > 0$, $i = 1, 2$.

Furthermore, the p.d.f's Φ_1, Φ_2 can be rewritten and arranged with respect to the three cases (Remark 2.1) in the following table:

Table (1)

p.d.f	$\alpha_1=\beta_1=\alpha_2=\beta_2=1$	$\alpha_1>\beta_1:\alpha_1=2,\beta_1=1;$ $\alpha_2<\beta_2:\alpha_2=1,\beta_2=2$	$\alpha_1<\beta_1:\alpha_1=1,\beta_1=2;$ $\alpha_2>\beta_2:\alpha_2=2,\beta_2=1$
$\Phi_1(x)$	$0.39903+0.95076\,e^{-x}$	$0.79807\,x+0.95076\,e^{-x}$	$0.79808-0.79808\,x+0.95076\,e^{-x}$
$\Phi_2(x)$	$0.35980+0.85727e^{-x^2}$	$0.71960-0.71960\,x+0.85727e^{-x^2}$	$0.71959-0.85727\,e^{-x^2}$

Figures 1 - 3 represent the graphs of three pairs of p.d.f's (Φ_1, Φ_2) with respect to the three cases (Remark 2.1).

The Figures

(1) $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$.

The p.d.f.'s:

$\Phi_1 = 0.39903 + 0.95076e^{-x}$

$\Phi_2 = 0.35980 + 0.85727e^{-x^2}$

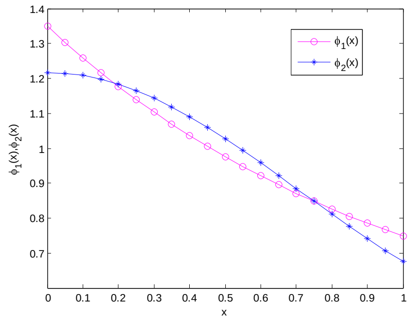

Fig.1

(2) $\alpha_1 > \beta_1 : \alpha_1 = 2,\ \beta_1 = 1;\ \alpha_2 < \beta_2 : \alpha_2 = 1,\ \beta_2 = 2$.

The p.d.f.'s:

$\Phi_1(x) = 0.79807\,x + 0.95076\ e^{-x}$

$\Phi_2(x) = 0.71960 - 0.71960\,x + 0.85727\,e^{-x^2}$

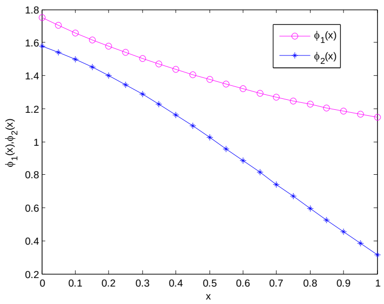

Fig. 2

(3) $\alpha_1 < \beta_1 : \alpha_1 = 1,\ \beta_1 = 2;\ \alpha_2 > \beta_2 : \alpha_2 = 2,\ \beta_2 = 1$.

The p.d.f.'s:

$\Phi_1(x) = 0.79808 - 0.79808\,x + 0.95076\,e^{-x}$

$\Phi_2(x) = 0.71959 - 0.85727\,e^{-x^2}$

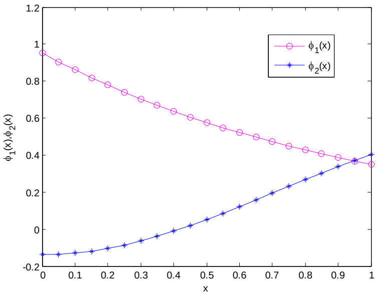

Fig. 3

3.1 Moments, Variances

By table (1) and three cases (Remark 2.1). It is easy to derive the first moments of each $\Phi_1(x)$ and $\Phi_2(x)$ by the formulas:

$$E_{\Phi_1}(x) = \int_0^1 x\,\Phi_1(x)dx\ ,$$

$$E_{\Phi_2}(x) = \int_0^1 x\Phi_2(x)dx$$

While, the second moments can also be derived by the formulas:

$$E_{\Phi_1}(x^2) = \int_0^1 x^2\Phi_1(x)dx,$$

$$E_{\Phi_2}(x^2) = \int_0^1 x^2\Phi_2(x)dx$$

and the calculations of the variances of the p.d.f's $\Phi_1(x)$, $\Phi_2(x)$ by the known formula $Var_{\Phi}(x) = E_{\Phi}(x^2) - [E_{\Phi}(x)]^2$ for the three cases (Remark 2.1) are tabulated in Table (2) which also contains the derived second moments.

Table (2)

p.d.f	$\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$			$\alpha_1 > \beta_1 : \alpha_1 = 2, \beta_1 = 1;$ $\alpha_2 < \beta_2 : \alpha_2 = 1, \beta_2 = 2$			$\alpha_1 < \beta_1 : \alpha_1 = 1, \beta_1 = 2;$ $\alpha_2 > \beta_2 : \alpha_2 = 2, \beta_2 = 1$		
	$E_\Phi(x)$	$E_\Phi(x^2)$	$var_\Phi(x)$	$E_\Phi(x)$	$E_\Phi(x^2)$	$var_\Phi(x)$	$E_\Phi(x)$	$E_\Phi(x^2)$	$var_\Phi(x)$
Φ_1(x)	0.65026	0.55172	0.12888	0.51725	0.35221	0.08466	0.78328	0.75125	0.13772
Φ_2(x)	0.63075	0.52222	0.12437	0.75068	0.70213	0.13861	−0.04109	0.01747	0.01578

3.2 Correlation Coefficients

By the known Pearson's correlation coefficient formula for dependent random variables X, Y:

$$\rho_{X,Y} = \frac{E(X,Y) - E(X)E(Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

and since each pair of any p.d.f's $\Phi_1(x)$, $\Phi_2(x)$ Table (1) with respect respectively to the three cases (Remark 2.1) are dependent. By writing:

E(X,Y) = expectation of the product $\Phi_1(x)$ by $\Phi_2(x)$

E(X)E(Y) = product of expectations of $\Phi_1(x)$ by $\Phi_2(x)$

Var(x) = variance of $\Phi_1(x)$

Var(Y) = variance of $\Phi_2(x)$

Then:

(i) For the case $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$

$$\rho_{\Phi_1,\Phi_2} = \frac{0.48781 - (0.65026)(0.63075)}{\sqrt{(0.12888)(0.12437)}} = 0.61357 \qquad \ldots(3.2.1)$$

(ii) For the case $\alpha_1 > \beta_1 : \alpha_1 = 2,\ \beta_1 = 1;\ \alpha_2 < \beta_2 : \alpha_2 = 1,\ \beta_2 = 2.$

$$\rho_{\Phi_1,\Phi_2} = \frac{0.39412 - (0.51725)(0.75068)}{\sqrt{(0.08466)(0.13861)}} = 0.05383 \qquad \ldots(3.2.2)$$

(iii) For the case $\alpha_1 < \beta_1 : \alpha_1 = 1,\ \beta_1 = 2;\ \alpha_2 > \beta_2 : \alpha_2 = 2,\ \beta_2 = 1$

$$\rho_{\Phi_1,\Phi_2} = \frac{-0.07424 - (0.78328)(-0.04109)}{\sqrt{(0.13772)(0.01578)}} = -0.90197 \qquad \ldots(3.2.3)$$

4- Conclusion and Recommendation

According to the three considering cases (Remark 2.1), the variances of the resulting p.d.f 's $\Phi_1(x)$, $\Phi_2(x)$ are almost equals with respect to case

1, while the variance of $\Phi_2(x)$ is greater than the variance of $\Phi_1(x)$ with respect to case 2 and with respect to case 3, the variance of $\Phi_1(x)$ is greater than the variance of $\Phi_2(x)$.

Furthermore, the correlation coefficient between the p.d.f 's $\Phi_1(x)$, $\Phi_2(x)$ is the greater under the case 1 ($\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$).

As a recommendation we suggest to consider other cases of the values of the parameters of the Beta distribution differ by the cases which studied in this article.

5- REFERENCE

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