Picard, Adomian and Predictor-Corrector methods for integral equations of fractional order

W.K.Zahra[1], M.A.Shehata[2]

[1]Faculty of Engineering, Tanta University, Tanta, Egypt

[2]Faculty of Engineering, Delta University for Science and Technology, Gamasa, Egypt

Abstract

In this paper, a comparative study of Picard method, Adomian method and Predictor-Corrector method are presented for fractional integral equation. In Picard method [6] a uniform convergent solution for the fractional integral equation is obtained. Also, for Adomian method, we construct a series solution see ([1], [5] and [7]). Finally, Predictor-Corrector method is used for solving fractional integral equation. Two test problems are discussed to compare the maximum error for each method.

Keywords

Integral equation, Picard method, Adomian method, Predictor-Corrector method, Continuous unique solution, Fractional-order integration, Convergence analysis, Error analysis.

Mathematical subject classification (2000)
26A33. 26A18. 39B12.

I- Introduction

Integral equations (IEs) are important in many applications such as radiative energy transfer and the oscillation of a string, membrane, or axle. In this paper, let $\alpha \in [0, 1]$, we study the existence and uniqueness of the fractional integral equation

$$x(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s)\big)ds, \quad \alpha \in R^+, \qquad (1)$$

And comparing the results obtained from the methods; Picard, ADM and Predictor-Corrector to obtain approximate solutions of the problem (1). Now, the definition of the fractional order integral operation is given by the following.

Definition1. Let β be a positive real number, the fractional-order integral of order β of the function f is defined on the interval [a, b] by (see kilbas 2006; podlubny 1999; Ross and Miller 1993)

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds,$$

And when a=0, we have $I^\beta f(t) = I_0^\beta f(t)$

II- Main Theorem

We discuss Eq. (1) in case of the following assumptions:

(i) $a: I \to R = [0, +\infty)$ is continuous on I where $I = [0,1]$;

(ii) $f: I \times R \to R$ is continuous and bounded with $M = sup_{(t,x)\in I} |f(t,x)|$

(iii) f satisfies Lipchitz condition with Lipchitz constant L such that $|f(t,x) - f(t,y)| \le L|x-y|$

Let $C = C(I)$ be the space of all real valued functions which are continuous on I.
Consider the operator F as

$$(Fx)(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s)\big)ds \ , \alpha > 0 \ \forall \ x \in c.$$

Theorem 1 Let the assumptions (i)-(iii) be satisfied. If $\frac{LT^\alpha}{\Gamma(\alpha+1)} < 1$, then the integral equation (1) has a unique solution $x \in C$.

Proof: See [14].

III- Method of successive approximations (Picard method)

Recall that Picard's method by Emile Picard in 1891 is used for the proof of existence and uniqueness of solutions of a system of differential equations. Applying Picard method to the problem (1), the solution is developed by the sequence

$$x_n(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x_{n-1}(s)\big)ds, n = 1,2, \dots$$

$$x_0(t) = a(t). \qquad (2)$$

All the functions $x_n(t)$ are continuous functions and x_n can be written as:

$$x_n = x_0 + \sum_{J=1}^{n} (x_j - x_{j-1}),$$

This means that convergence of the sequence x_n is equivalent to the convergence of the infinite series $\sum(x_j - x_{j-1})$ and the solution will be

$$x(t) = \lim_{n\to\infty} x_n(t),$$

i.e., if the infinite series $\sum(x_j - x_{j-1})$ converges, then the sequence $x_n(t)$ will converge to $x(t)$. To prove the uniform convergence of $\{x_n(t)\}$, we shall consider the associated series

$$\sum_{n=1}^{\infty} [x_n(t) - x_{n-1}(t)].$$

From (2) for n=1, we get

$$x_1(t) - x_0(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_0(s))ds,$$

And

$$|x_1(t) - x_0(t)| \le M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$\le M \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$\le M \frac{T^\alpha}{\Gamma(\alpha+1)}. \quad (3)$$

Now, we shall obtain an estimate for $x_n(t) - x_{n-1}(t), n \ge 2$

$$x_n(t) - x_{n-1}(t) \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{n-1}(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{n-2}(s))ds$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [f(s, x_{n-1}(s)) - f(s, x_{n-2}(s))]ds$$

Using the assumptions (ii) and (iii) we get

$$|x_n(t) - x_{n-1}(t)| \le L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_{n-1}(s) - x_{n-2}(s)|\, ds.$$

Putting n=2, then using (3), we get

$$|x_2(t) - x_1(t)| \le L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_1(s) - x_0(s)|\, ds$$

$$\le ML \frac{T^\alpha}{\Gamma(\alpha+1)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds.$$

$$|x_2 - x_1| \le ML \frac{T^\alpha}{\Gamma(\alpha+1)} \frac{T^\alpha}{\Gamma(\alpha+1)} \le ML \frac{T^{2\alpha}}{\Gamma(\alpha+1)} \le ML\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2,$$

$$|x_3(t) - x_2(t)| \le L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_2(s) - x1s\, ds$$

$$\le MLL\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$\le MLL\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 \frac{T^\alpha}{\Gamma(\alpha+1)} \le \frac{M}{L}\left(\frac{LT^\alpha}{\Gamma(\alpha+1)}\right)^3$$

By the same manner, we obtain the general estimate as:

$$|x_n - x_{n-1}| \le \frac{M}{L}\left(\frac{LT^\alpha}{\Gamma(\alpha+1)}\right)^n.$$

Since $\left(\frac{LT^\alpha}{\Gamma(\alpha+1)}\right) < 1$ then the uniform convergence of

$$\sum_{n=1}^{\infty} [x_n(t) - x_{n-1}(t)]$$

Is proved and so, the sequence $\{x_n(t)\}$ is uniformly convergent.

Since $f(t, x)$ is continuous in x then

$$x(t) = \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(s))ds$$

$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds$$

Thus, a unique solution of (1) is achieved.

IV- Adomian decomposition method (ADM)

The Adomian decomposition method (ADM) is anon-numerical method for solving wide variety of functional equations and usually gets the solution in series form. Since the beginning of the 1980s, Adomian, see ([1]-[5], and [7]) has presented and devolved a so-called decomposition method for solving algebraic, differential, integro-differential, differential with delay, and partial differential equations.

The solution is presented as an infinite series which converges rapidly to the accurate solutions. The method has many merits over other methods, mainly, this means that there is no need to linearization, perturbation which may change the nature of the problem being solved and the method has many applications in applied sciences. For more details, see ([1]-[5], and [7]).

In this section, Adomian decomposition method (ADM) will be applied for the integral equation (1)

The solution of the integral equation (1) using ADM is:

$$x_0(t) = a(t) \quad (4)$$

$$x_i(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A_{i-1}(s)ds, \quad (5)$$

Where A_i is adomian polynomial of the non –linear term of $f(s, x)$ which has the form:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [f(t, \sum_{i=0}^{\infty} \lambda^i x_i)]_{\lambda=0}, \quad (6)$$

And the solution will be

$$x(t) = \sum_{i=1}^{\infty}(x_i\,(t)\,). \qquad (7)$$

A- Convergence analysis

Theorem 2. Let the solution of the FIE (1) be exist. If $|x_1(t)| < k, k$ is appositive constant, then the series solution (7) of the FIE (1) using ADM converges.

Proof. Let the sequence $\{Sp\}$ such that $Sp = \sum_{i=0}^{p} x_i\,(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty}(x_i\,(t)\,)$ and we have

$$f(t,x) = \sum_{i=0}^{\infty} A_i.$$

Let Sp and Sq be two arbitrary partial sums with $p > q$.

Now, we will prove that $\{Sp\}$is Cauchy sequence in the Banach- space.

$$Sp - Sq = \sum_{i=0}^{p} x_i \; - \sum_{i=0}^{q} x_i$$

$$= (\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} A_{i-1}(s)\; ds\;) - (\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{q} A_{i-1}(s)\; ds\;)$$

$$= (\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\,\sum_{i=0}^{p} A_{i-1}(s) - \sum_{i=0}^{q} A_{i-1}(s)]\, ds\,)$$

$$\|S_p - S_q\| \leq max_{t\in I}\left|(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=q+1}^{p} A_{i-1}(s)\,ds\;)\right|$$

$$\leq max_{t\in I}\left|(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=q}^{p-1} A_i(s)\; ds\;)\right|$$

$$\leq max_{t\in I} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \;|f(\tau,S_{p-1}) - f(\tau,S_{q-1})|\, d\tau$$

$$\leq max_{t\in I} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \;|(S_{p-1}) - (S_{q-1})|\; d\tau$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \left[\frac{LT^\alpha}{\Gamma(\alpha+1)}\right] max_{t\in I}|(S_{p-1}) - (S_{q-1})\;|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \left[\frac{LT^\alpha}{\Gamma(\alpha+1)}\right] \|(S_{p-1}) - (S_{q-1})\|$$

$$\leq \beta\|(S_{p-1}) - (S_{q-1})\|$$

Let $p = q + 1$ then

$$\|S_{q+1} - S_q\| \leq \beta\|S_q - S_{q-1}\| \leq \beta^2\|S_{q-1} - S_{q-2}\| \leq \cdots \leq \beta^q\|S_1 - S_0\|$$

From the triangle inequality, we have

$$\|S_p - S_q\| \leq \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \cdots \|S_p - S_{p-1}\|.$$

$$\leq [\beta^q + \beta^{q+1} + \cdots \ldots\ldots\ldots + \beta^{p-1}]\; \|S_1 - S_0\|$$

$$\leq \beta^q[1 + \beta + \cdots \ldots\ldots\ldots + \beta^{p-q-1}]\; \|S_1 - S_0\|$$

$$\leq \beta[\frac{1-\beta^{p-q}}{1-\beta}]\; \|x_1\|$$

Now, $0 < \beta < 1$,and $p > q$ implies that

$1 - \beta^{p-q} \leq 1$ Consequently,

$$\|S_p - S_q\| \leq \frac{\beta^q}{1-\beta}\|x_1\| \;\leq \frac{\beta^q}{1-\beta} max_{t\in I}|x_1|$$

But $|x_1| < k$ and as $q \to \infty$, then $\|S_p - S_q\| \to 0$ and hence, $\{Sp\}$ is Cauchy sequence in the banach space E and the series $\sum_{i=0}^{\infty}(x_i\,(t)\,)$ converges.

B-Error analysis

Theorem 3. The maximum truncate error of the series solution (7) to the problem (1) is estimated by

$$max_{t\in I}\left|x(t) - \sum_{i=0}^{q} x_i\,(t)\right| \leq \frac{\beta^q}{1-\beta} max_{t\in I}|x_1(t)|$$

Proof. From theorem 2, we have

$$\|S_p - S_q\| \;\leq \frac{\beta^q}{1-\beta} max_{t\in I}|x_1(t)|$$

But $Sp = \sum_{i=0}^{p} x_i\,(t)$ as $p \to \infty$, then $S_p \to x(t)$ so,

$$\|x(t) - S_q\| \;\leq \frac{\beta^q}{1-\beta} max_{t\in I}|x_1(t)|$$

So, the maximum error in the interval I is

$$max_{t\in I}|x(t) - \sum_{i=0}^{q} x_i\,(t)| \leq \frac{\beta^q}{1-\beta} max_{t\in I}|x_1(t)|$$

And this completes the proof.

V- predictor-corrector method

An Adams-type-Predictor Corrector method has been introduced see ([2]-[3]).In this section, we use an Adams-type-Predictor Corrector method for integral eq. (1) .The product trapezoidal quadrature formula is used t_j $(j = 0,1,\ldots\ldots\ldots, k+1)$ taken with respect to the weight function $(t_{k+1} - \,.)^{\alpha-1}$.In other words, one applies the approximation:

$$\int_{t_0}^{t_{k+1}} (t_{k+1} - \;u)^{\alpha-1}\, g(u)du \;\approx\; \int_{t_0}^{t_{k+1}} (t_{k+1} - u)^{\alpha-1} \tilde{g}_{k+1}(u)du$$

$$= \sum_{j=0}^{k+1} \tilde{a}_{j,k+1}\; g(t_j),$$

Where:

$$\tilde{a}_{j,k+1} = \begin{cases} \frac{h^{\alpha}}{\alpha(\alpha+1)} \left[\begin{matrix} k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha} & ,j=0 \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1} & ,1 \le j \le k \\ 1 & ,j = k+1 \end{matrix} \right] \end{cases}$$

And h is a step size, this yields the corrector-formula, i.e. the fractional variant of the one-step Adams moulton method .The corrector formula is:

$$x_{k+1} = a(t_{k+1}) + \frac{1}{\Gamma(\alpha)} \left[\sum_{j=0}^{k} \tilde{a}_{j,k+1} f(t_j, x(t_j)) + \tilde{a}_{k+1,k+1} f(t_{k+1}, x_{k+1}^{p}) \right]$$

Now, we need to determine the predictor formula by calculating the term ${x_{k+1}}^{p}$. The integral on the right hand side of equation (1) is replaced by following rule, i.e.:

$$\int_{t_0}^{t_{k+1}} (t_{k+1} - u)^{\alpha-1} g(u) du \approx \sum_{j=0}^{k} b_{j,k+1}\, g(t_j),$$

Where:

$$b_{j,k+1} = \frac{h^{\alpha}}{\alpha} [(k+1-j)^{\alpha} - (k-j)^{\alpha}]$$

Thus, the predictor ${x_{k+1}}^{p}$ is determined by the fractional Adams-Bash forth method:

$${x_{k+1}}^{p} = a(t_{k+1}) + \frac{1}{\Gamma(\alpha)} \left[\sum_{j=0}^{k} b_{j,k+1} f(t_j, x(t_j)) \right].$$

VI- Numerical Examples

In this section, two numerical examples are solved by Picard, ADM and predictor-corrector methods and a comparisons between these methods are illustrated.

Example 1. Consider the following FIE,

$$x(t) = t^2 - \frac{\Gamma(5) t^{4.98}}{5\Gamma(5.98)} + \frac{1}{5} \int_0^t \frac{(t-s)^{-0.02}}{\Gamma(0.98)} x^2(s)\, ds, \quad (8)$$

Where the exact solution is $x(t) = t^2$.

Applying Picard method to Eq. (8), we get

$$x_n(t) = \left(t^2 - \frac{\Gamma(5) t^{4.98}}{5\Gamma(5.98)} \right) + \frac{1}{5} \int_0^t \frac{(t-s)^{-0.02}}{\Gamma(0.98)} {x_{n-1}}^2(s) ds, n = 1,2, \dots \dots \dots$$

$$x_0(t) = \left(t^2 - \frac{\Gamma(5) t^{4.98}}{5\Gamma(5.98)} \right),$$

And the solution will be

$$x(t) = x_n(t).$$

Applying ADM to eq. (8), we get

$$x_0(t) = \left(t^2 - \frac{\Gamma(5) t^{4.98}}{5\Gamma(5.98)} \right)$$

$$x_i(t) = \frac{1}{5} \int_0^t \frac{(t-s)^{-0.02}}{\Gamma(0.98)} A_{i-1}(s)\, ds, i \ge 1$$

Where A_i is the adomian polynomial of the non – linear term x^2, and the solution will be

$$x(t) = \sum_{i=0}^{q} (x_i(t))$$

Applying PECE to equation (8), we have

$$x_0 = 0, f(t,x) = x^2$$

Table 1 .Shows a comparison between the absolute error of Picard (when $n = 3$), ADM solutions (when $q = 3$) and PECE solutions.

Table 1: Absolute Error

T	$\lvert x_{exact} - x_{picard} \rvert$	$\lvert x_{exact} - x_{ADM} \rvert$	$\lvert x_{exact} - x_{PECE} \rvert$
0.1	9.5259×10^{-1}	4.20747×10^{-2}	4.96261×10^{-6}
0.2	1.87127×10^{-12}	5.14305×10^{-1}	0.0000746928
0.3	1.57933×10^{-1}	4.85964×10^{-1}	0.000368798
0.4	3.67373×10^{-9}	6.26596×10^{-1}	0.00115021
0.5	4.21676×10^{-8}	2.71015×10^{-12}	0.00278627
0.6	3.09553×10^{-7}	5.86835×10^{-11}	0.00575272
0.7	1.66897×10^{-6}	7.87483×10^{-1}	0.0106424
0.8	7.17716×10^{-6}	7.43734×10^{-9}	0.0181816
0.9	0.0000259625	5.36563×10^{-8}	0.0292567
1	0.0000819176	3.12665×10^{-7}	0.0449573

Example 2 .Consider the following FIE,

$$x(t) = t - \frac{\Gamma(4) t^{3.95}}{7\Gamma(4.95)} + \frac{1}{7} \int_0^t \frac{(t-s)^{-0.05}}{\Gamma(0.95)} x^3(s)\, ds, \quad (9)$$

Where the exact solution is $x(t) = t$.

Applying Picard method to eq. (9), we get

$$x_n(t) = \left(t - \frac{\Gamma(4) t^{3.95}}{7\Gamma(4.95)} \right) + \frac{1}{7} \int_0^t \frac{(t-s)^{-0.05}}{\Gamma(0.95)} {x_{n-1}}^3(s) ds, \quad n = 1,2, \dots$$

$$x_0(t) = \left(t - \frac{\Gamma(4) t^{3.95}}{7\Gamma(4.95)} \right),$$

And the solution will be

$$x(t) = x_n(t)$$

Applying ADM to eq. (9), we get

$$x_0(t) = \left(t - \frac{\Gamma(4)t^{3.95}}{7\Gamma(4.95)}\right)$$

$$x_i(t) = \frac{1}{7}\int_0^t \frac{(t-s)^{-0.05}}{\Gamma(0.95)} A_{i-1}(s)\, ds\, , i \geq 1.$$

Where A_i is the adomian polynomial of the non – linear term x^3, and the solution will be

$$x(t) = \sum_{i=0}^{q} (x_i\,(t)\,)$$

Applying PECE to equation (9), we have

$$x_0 = 0\, , f(t,x) = x^3$$

Table 2 .Shows a comparison between the absolute error of Picard (when $n = 3$), ADM solutions (when $q = 3$) and PECE solutions.

Table 2: Absolute Error

t	$\lvert x_{exact} - x_{picard}\rvert$	$\lvert x_{exact} - x_{ADM}\rvert$	$\lvert x_{exact} - x_{PECE}\rvert$
0.1	1.80385×10^{-1}	1.01227×10^{-21}	0.0000562405
0.2	1.66447×10^{-11}	1.28764×10^{-1}	0.00043308
0.3	9.02793×10^{-1}	7.61601×10^{-1}	0.00144209
0.4	1.53418×10^{-8}	7.03722×10^{-13}	0.00339566
0.5	1.3799×10^{-7}	2.34585×10^{-11}	0.00661179
0.6	8.29663×10^{-7}	4.09573×10^{-1}	0.0114232
0.7	3.77646×10^{-6}	4.56791×10^{-9}	0.0181922
0.8	0.0000140161	3.66269×10^{-8}	0.0273329
0.9	0.0000444918	2.27818×10^{-7}	0.0393455
1	0.000124786	1.15748×10^{-6}	0.0548674

Conclusion

In this paper, Picard, ADM and Predictor-Corrector are presented for solving fractional integral equation. These methods have been analyzed. Two numerical examples are devoted to compare the maximum absolute error for the theses methods. We can conclude from the above examples that ADM gives more accurate solution than the other two methods.

References

[1] G. Adomian, Stochastic System, Academic press, 1983.

[2] K Diethelm, A Freed, TheFrac.PECE subroutine for the numerical solution of differential equations of fractional order, in: S Heinzel, T Plesser (Eds.), Forschung and wissens chaftliches Rechnen 1998, Gesellschaftf AEur Wisses chaftliche Datenverar beitung, G"ottingen, 1999, pp 57-71.

[3] Diethelm, N J Ford and A D Freed. Predictor Corrector approach For the numerical solution of fractional differential equations, Nonlinear Dynamics, 29, 3- 22, 2002.

[4] K. Abbaoui, Y. Cherruault, Convergence of Adomian's method Applied to Differential Equations, Computers Math. Applic. 28 (1994) 103-109.

[5] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1995.

[6] N. Bellomo and D. Sarafyan, On Adomian's decomposition method and some comparisons with Picard's iterative scheme, Journal of Mathematical Analysis and Applications 123 (1987) 389-400..

[7] Y. Cherruault, Convergence of Adomian method, Kybernetes, 18 (1989) 31-38.

[8] C. Corduneanu, Principles of Differential and integral equations, Allyn and Bacon. Hnc, New Yourk, 1971.

[9] A. M. A. El-Sayed, H. H. G. Hashem, Integrable and continuous solutions of nonlinear quadratic integral equation, Electronic Journal of Qualitative Theory of Differential Equations 25 (2008) 1-10.

[10] A. M. A. El-Sayed, H. H. G. Hashem, Monotonic positive solution of nonlinear quadratic Hammerstein and Urysohn functional integral equations, Commentationes Mathematical. 48 (2) (2008) 199-207

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, North-Holland, 2006

[12] I. Podlubny, Fractional Differential equations. San Diego-New York-London, 1999.

[13] B. Ross, K. S. Miller, an Introduction to Fractional Calculus and Fractional Differential Equations. John Wiley, New York, 1993