On The Solvability of Nonlinear Integral Functional Equation

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Abstract : We study the existence solution of a functional Volterra integral equation in the space of Lebesgueintegrable functions on an unbounded interval by using the Schauder fixed point theorem and weak measure of noncompactness.

Keywords : Superposition operator- Caratheodory conditions- Nonlinear integral functional equation- Measure of weak noncompactness- Schauder fixed point theorem.

Mathematics Subject Classification :43H15, 24A06, 45M27, 18K05 .

1. Introduction

The subject of nonlinear integral equation considered as an important branch of mathematics because it is used for solving of many problems such as physics, engineering and

economics [29,31].

In this paper, we will investigate the solvability of the functional Volterra integral equation

$$x(t) = g(t)\, f(t, x(t)) + h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big) ds, \qquad (1.1)$$

which is the general form of other functional integral equations (cf. [6,25]). In [24], this equation had been investigated befor where the existence of monotonic solutions were proved.

2. Preliminaries

Let $\mathbb{R}$be the field of real number, $\mathbb{R}^+$be the interval $[0,\infty)$. If A is a Lebesgue mea-surable subset of $\mathbb{R}$**,** then the symbol meas(A) stands for the Lebesgue measure of A.

Further, denote by $L_1(A)$ the space of all real functions defined and Lebesgue mesurable on the set A. The norm of a function $x \in L_1(A)$is defined in the standard way by the formula,

$$\|x\| = \|x_{L_1(A)}\| = \int_A |x(t)| dt \qquad (2.1)$$

Obviously $L_1(A)$ forms a Banach space under this norm. The space $L_1(A)$ will be called the lebesgue space. In the case when $A = \mathbb{R}^+$ we will write L_1 instead of $L_1(\mathbb{R}^+)$.

One of the most important operator studied in nonlinear functional analysis is the so called the superposition operator [1]. Now, let us assume that $I \subset \mathbb{R}$is a given interval, bounded or not.

Definition 2.1 [28] Assume that a function $f(t,x) = f : I \times \mathbb{R} \to \mathbb{R}$satisfiesthe caratheodory conditions, i.e. it is measurable in t for any $x \in \mathbb{R}$and continuous in $x$for almost all $t \in I$. Then to every function x being measurableon Iwe may assign $F_f(x)(t) = f\big(t, x(t)\big), t \in I$.

The operator $F_f$defined in such a way is called the superposition (Nemytskii) operator generated by the function f.

We have the following theorem.

Theorem 2.1 [1] Suppose that f satisfiesthecaratheodory conditions. The superposition operator F maps continuously the space $L_1(I)$ into $L_1(I)$if and only if

$|f(t,x)| \leq a(t) + b|x|,$ (2.2) for $t \in I$ and $x \in \mathbb{R}$,
where $a \in L_1(I)$ and $b \geq 0$.

This theorem was proved by Krasnoselskii [1] in the case when I is bounded interval. The generalization to the case of an unbounded interval I was given by Appell and Zabrejko [1].

Let I be an interval, in the following two theorems " Lusin and Dragoni " [12,29], which explain the structure of measurable functions and functions satisfying caratheodory condition, where D^c denotes the complement of D.

Theorem 2.2 Let$m : I \to \mathbb{R}$ be a measurable function. For any $\varepsilon > 0$ there exist a closed subset D_ε of the interval I such that meas$(D_\varepsilon^c) \leq \varepsilon$ and $m|_{D_\varepsilon}$ is continuous **Theorem 2.3** Let $f : I \times \mathbb{R} \to \mathbb{R}$be a function satisfingthecaratheodory conditions. Then for each $\varepsilon > 0$ there exist a closed subset D_ε of the interval I such that meas$(D_\varepsilon^c) \leq \varepsilon$ and $f|_{D_\varepsilon \times \mathbb{R}}$ is continuous.

Now we present the concept of measure of weak noncompactness. Assume that $(E, \|.\|)$ is an arbitrary Banach space with zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol $B_r$stands for the ball $B(\theta,r)$.
Denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E^W$ its subfamily consisting of all relatively weakly compact sets. The symbol $\overline{X}^W$ stands for the weak closure of a set X and the symbol conv X will denote the convex closed hull of a set X.Now we present the following definition [28].
Definition 2.2 [28] A mapping $\mu : \mathcal{M}_E \to [0,\infty)$ is said to be a measure of weak noncompactness in E if it satisfies the following conditions :
i) The family ker$\mu = \{ X \in \mathcal{M}_E : \mu(X) = 0 \}$ is nonempty and $\beta \subset \mathcal{N}_E^W$.
ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
iii) $\mu(convX) \leq \mu(X)$,
iv) $\mu\big(\lambda X + (1-\lambda)\big) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)for\lambda \in [0,1]$.
v) If $X_n \in \mathcal{M}_E$ and $X_n = \overline{X}^W{}_n$, and$X_{n+1} \subset X_n$, for $n = 1,2,\dots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$.
Then the intersection $X_\infty = \cap_{n=1}^{\infty} X$ is nonempty.
It is worthwhile mentioning that the first important example of measure of weak noncompactness has been defined by De-Blasi [10] in the following way :
$\beta(X)$ =inf{ $r > 0$: there exist a weakly compact subset W of E such that $x \subset W + B_r$ }.
Also, we recall the following criterion for weak noncompactness due to Dieudonne [11], which is of fundamental importance in our subsequent analysis.
Theorem 2.4 [7] Abounded set X is relatively weakly compact in L_1 if and only if the following two conditions are satisfied
a) for any $\varepsilon > 0$ there exist $\delta > 0$ such that if meas$(D) < \delta$ then
$\int_D |x(t)|dt \leq \varepsilon$for all $x \in X$.
b) for any $\varepsilon > 0$ there exist $T > 0$ such that $\int_T^\infty |x(t)|dt \leq \varepsilon$ for all $x \in X$. Now, for nonempty and bounded subset X of the space L_1 let us define :

$$c(X) = \lim_{\varepsilon\to 0} \sup_{x\epsilon X}\{\sup\{\textstyle\int_D |x(t)|dt : D \subset R^+,\ \text{meas(D)} \leq \varepsilon\} \quad (2.3)$$

And

$$d(X) = \lim_{T\to\infty} \sup\{\textstyle\int_T^\infty |x(t)|dt : x \in X\}. \quad (2.4)$$

put

$$\gamma(X) = c(X) + d(X) \quad (2.5)$$

Where the function γ is a measure of weak noncompactness in the space L_1. For any nonempty and bounded subset $X$of $L_1(\mathbb{R}^+)$and we have the following theorem.

Theorem 2.5 [7] The function γ is a regular measure of weak noncompactness in the space $L_1(\mathbb{R}^+)$ such that

$$\beta(X) \leq \gamma(X) \leq 2\beta(X).$$

Also, we have the following theorem.

Theorem 2.6 [26] (Schauder fixed point theorem).

Assume that X is a nonempty, convex, closed and bounded subset of a Banach space E and $G : E \to E$ is completely continuous mapping (i.e.G is continuous and $G(Y)$ is relatively compact for every bounded subset Y of E) such that

$G: X \to X$. Then G has at least a fixed point in X.

3. ExistenceTheorem

Define the operator H associated with integral Equation (1.1) take the following form.

$$Hx = Ax + Bx. \quad (3.1)$$

Where

$(Ax)(t) = g(t)\, f(t, x(t))$,

$(Bx)(t) = h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big) ds$

$= h(t) + KFx(t)$,

where $(Kx)(t) = \int_0^t k(t,s)\, x(s)\, ds$,

$Fx = f(t,x)$,are linear operator at superposition respectively.

We shall treat the equation (3.1) under the following assumptions listed below.

i) Let g $:\mathbb{R}^+ \to R$ be bounded function such that $M = \sup_{t\in\mathbb{R}^+} |g(t)|$,

$h : \mathbb{R}^+ \to R$, $h \in L_1(\mathbb{R}^+)$.

ii) $:\mathbb{R}^+ \to R$ satisfiesthe caratheodory conditions and there are positive function $a \in L_1$ and constant $b \geq 0$ such that :

$|f(t,x)| \leq a(t) + b|x|$.

iii) Assume $k : \mathbb{R}^+ \times \mathbb{R}^+ \to R$satisfiescaratheodory conditions such that the linear operator K defined as

$$(Kx)(t) = \int_0^t k(t,s)\, x(s)\, ds, \qquad t > 0 \quad (3.2)$$

maps the space L_1 into itself (note that due to this assumptions and [29] the linear operator K will be continuous whose norm $\|K\|$).

iv) q = $bM + b\|K\| < 1$.

Then we can prove the following theorem.

Theorem 3.1 Assume that the assumptions(i)-(iv) are satisfied, then equation (3.1) has at leastone integrable solution on $L_1(\mathbb{R}^+)$.

Proof : First, we will prove that the operator H maps continuously $L_1(\mathbb{R}^+)$ into $L_1(\mathbb{R}^+)$.

$$\int_0^\infty |(Hx)(t)| dt = \int_0^\infty \left| g(t)\, f\big(t, x(t)\big) + h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big) ds \right| dt$$

$$\leq \int_0^\infty \left| g(t)\, f\big(t, x(t)\big) \right| dt + \int_0^\infty \left| h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big) ds \right| dt \leq \int_0^\infty |g(t)| [a(t) + b|x(t)|] dt + \int_0^\infty |h(t)| dt$$

$$+ \|K\| [a(s) + b|x(s)|] ds$$

$$\leq M\|a\| + bM \int_0^\infty |x(t)| dt + \|h\| + \|K\|\|a\| + b\|K\| \int_0^\infty |x(s)| ds$$

$$\leq [M + \|K\|]\|a\| + \|h\| + [bM + b\|K\|] \int_0^\infty |x(t)| dt < \infty$$

Then due to assumptions (i), (ii), (iii) and theorem (2.1), we see that

$H : L_1(\mathbb{R}^+) \to L_1(\mathbb{R}^+)$continuously

Next, let x be an arbitrary function in the ball $B_r \subset L_1(\mathbb{R}^+)$. In view of our assumptions we get

$$\|Hx\| = \int_0^\infty \left| g(t)\, f\big(t, x(t)\big) + h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big) ds \right| dt$$

$$= \|gF\| + \|h\| + \|KFx\|$$

$\leq |g| \|F\| + \|h\| + \|K\| \|Fx\|$
$\leq M \int_0^\infty |f(t,x(t))| dt + \|h\| + \|K\| \int_0^\infty |f(t,x(t))| dt$
$\leq \|h\| + M \int_0^\infty [a(t) + b|x(t)|] dt + \|K\| \int_0^\infty [a(t) + b|x(t)|] dt$
$\leq \|h\| + M\|a\| + bM \int_0^\infty |x(t)|] dt + \|K\|\|a\| + b\|K\| \int_0^\infty |x(t)|] dt$
$\leq \|h\| + [M + \|K\|]\|a\| + [bM + b\|K\|]\|x\|$
Then
$\|Hx\| \leq \|h\| + [M + \|K\|]\|a\| + [bM + b\|K\|] r \leq r$
So, H transform B_r into B_r, where $r \leq \frac{\|h\| + [M + \|K\|]\|a\|}{1 - [bM + b\|K\|]}$.
Using assumption (iv), we see that $r > 0$.
Now, we will prove that $\beta(HX) \leq q\beta(X)$ for all bounded subset X of B_r.
Take an arbitrary number $\varepsilon > 0$ and a set $D \subset \mathbb{R}^+$ such that $\text{meas}(D) \leq \varepsilon$.
For any $x \in X$, we get

$$\int_D |(Hx)(t)| dt = \int_D \left| g(t) f(t,x(t)) + h(t) + \int_0^t k(t,s) f(s,x(s)) ds \right| dt$$
$$\leq \int_D |g(t) f(t,x(t))| dt + \int_D \left| h(t) + \int_0^t k(t,s) f(s,x(s)) ds \right| dt$$
$$\leq \int_D |g(t)||f(t,x(t))| dt + \int_D |h(t)| dt + \|KFx\|_{L_1(D)}$$
$$\leq M \int_D [a(t) + b|x(t)|] dt + \int_D |h(t)| dt + \|K\|_{L_1(D)} \int_D |f(t,x(s))| ds$$
$$\leq M \int_D a(t) dt + bM \int_D |x(t)| dt + \int_D |h(t)| dt + \|K\|_{L_1(D)} \int_D a(s) ds + b\|K\|_{L_1(D)} \int_D |x(t)| dt$$
$$\leq M \int_D a(t) dt + \int_D |h(t)| dt + [bM + b\|K\|_{L_1(D)}] \int_D |x(t)| dt$$

Where $\|K\|_{L_1(D)}$ denotes the norm of the operator $K : L_{1(D)} \to L_{1(D)}$.
Hence $\lim_{\varepsilon \to 0} \{\sup \{\int_D |h(t)| dt : D \subset R^+, \ \text{meas(D)} \leq \varepsilon\}\} =$
$= \limsup_{\varepsilon \to 0} [\int_D a(t) dt : D \subset R^+, \ \text{meas(D)} \leq \varepsilon\}] = 0$
We get

$$c(HX) \leq [\, q = bM + b\|K\|_{L_1(D)}]\, c(X) \qquad (3.3)$$

From $T > 0$, any $x \in X$, we have

$$\int_T^\infty |(Hx)(t)| dt = \int_T^\infty \left| g(t) f(t,x(t)) + h(t) + \int_0^t k(t,s) f(s,x(s)) ds \right| dt$$
$$\leq \int_T^\infty |g(t) f(t,x(t))| dt + \int_T^\infty \left| h(t) + \int_0^t k(t,s) f(s,x(s)) ds \right| dt$$
$$\leq \int_T^\infty |g(t)||f(t,x(t))| dt + \int_T^\infty |h(t)| dt + \int_T^\infty \left| \int_0^t k(t,s) f(s,x(s)) ds \right| dt \leq M \int_T^\infty [a(t) + b|x(t)|] dt$$
$$+ \|h\| + \|K\| \int_T^\infty [a(s) + b|x(s)|] ds$$
$$\leq M\|a\| + bM\|x\| + \|h\| + \|K\|\|a\| + b\|K\|\|x\|$$
$$\leq [bM + b\|K\|]\, \|x\|$$

By equation (2.4) we get

$$d(HX) \leq [\, q = bM + b\|K\|]\, d(X) \qquad (3.4)$$

By combining equations (3.3), (3.4) and equation (2.5), then using theorem (2.5), we deduce that $\beta(HX) \leq q\beta(X)$.

Now, we will construct a nonempty closedconvex weakly compact set Y on which we will apply a fixed point theorem, to do this. Let $B_r^1 = Conv(HB_r)$, where $Conv(HB_r)$ denotes the closure of the convex hull of HB_r, since $HB_r \subset B_r$, then $B_r^1 \subset B_r$ Similarly, let $B_r^2 = Conv(HB_r^1)$, then $B_r^2 \subset B_r^1$, also $B_r^3 = Conv(HB_r^2) \subset B_r^2$ and so on we get a decreasing sequence (B_r^n) of bounded, convex, closed subsets of B_r such that
$H(B_r^n) \subset B_r^n, \quad n \in N.$
Using the properties of the De-Blasi measure β of weak noncompactness, we see that

$$\beta(B_r^{n+1}) = \beta(ConvHB_r^n)$$
$$= \beta(HB_r^n)$$

$$\leq q\beta(B_r^n), n \geq N,$$

and so on, we have

$\beta(B_r^{n+1}) \leq q^n \beta(B_r^n), \quad q < 1, n \geq N.$

Hence, as $n \to \infty$, we get $\lim_{n\to\infty} \beta(B_r^{n+1}) = 0$.

So, $Y = \bigcap_{n \in N} B_r^n$ is a nonempty, closed, bounded convex and relatively weakly compact subset of B_r and $H(Y) \subset Y$.

In the sequel, we will prove that $H(Y)$ is relatively compact in the space L_1, to do this.

Let $\{y_n\}$ be a sequence in $Y$and > 0, then by using theorem (2.3) there exists a closed measurable subset D_ε of $[0,t]$ such that $m([0,t]/D_\varepsilon) < \varepsilon$ and $f|_{D_{\varepsilon \times R}}$ and $k|_{D_{\varepsilon \times R}}$ are continuous.

Let us take arbitrary $t_1, t_2 \in D_\varepsilon$and assume $t_1 < t_2$. For an arbitrary fixed $n \in N$.

Let $U_n(t) = \int_0^t k(t,s) f(s, y_n(s)) ds$.

Then, we have

$$|U_n(t_2) - U_n(t_1)| = \left|\int_0^{t_2} k(t_2,s) f(s,y_n(s))ds - \int_0^{t_1} k(t_1,s) f(s,y_n(s))ds\right|$$

$$= \left|\int_0^{t_2} k(t_2,s) f(s,y_n(s))ds - \int_0^{t_2} k(t_1,s) f(s,y_n(s))ds + \right.$$

$$\left. + \int_0^{t_2} k(t_1,s) f(s,y_n(s))ds - \int_0^{t_1} k(t_1,s) f(s,y_n(s))ds\right|$$

$$\leq \left|\int_0^{t_2} k(t_2,s) f(s,y_n(s))ds - \int_0^{t_2} k(t_1,s) f(s,y_n(s))ds\right| +$$

$$+ \left|\int_0^{t_2} k(t_1,s) f(s,y_n(s))ds - \int_0^{t_1} k(t_1,s) f(s,y_n(s))ds\right|$$

$$\leq \int_0^{t_2} |k(t_2,s) - k(t_1,s)| \; |f(s,y_n(s))| ds$$

$$+ \int_{t_1}^{t_2} |k(t_1,s)| \; |f(s,y_n(s))| ds$$

$$\leq \int_0^{t_2} |k(t_2,s) - k(t_1,s)| \; [a(t) + b|y_n(s)|] ds$$

$$+ \int_{t_1}^{t_2} |k(t_1,s)| \, [a(t) + b|y_n(s)|] \, ds$$

We infer that the number $t_2 - t_1$ is small enough, then the right hand side of last equation tends to zero independently y_n as $(t_2 - t_1) \to 0$.

Since $\{y_n\} \subset Y$ and Y is bounded, then $\{y_n\}$ is bounded. Hence (U_n) is a sequence of equicontinuous and uniformly bounded function in $C(D_\varepsilon)$ and so $\{B(y_n)\}$ is a sequence of equicontinuous and uniformly bounded function in (D_ε) . Also hence $\{A(y_n)\}$ is a sequence of equicontinuous and uniformly bounded function in $C(D_\varepsilon)$. Then $\{H(y_n)\}$ is a sequence of equicontinuous and uniformly bounded function in (D_ε) . Now by using Ascoli-Arzela theorem, we deduce that $\{H(y_n)\}$ is relatively compact in $C(D_\varepsilon)$, from which, we deduce that $\{H(y_n)\}$ is Cauchy sequence in $C(D_\varepsilon)$.

Next , we will use the last result to prove that $\{H(y_n)\}$ is Cauchy sequence in L_1. Using theorem (2.4) and the fact $H(Y)$ is relatively compact in $C(D_\varepsilon)$ that proved before in our theorem, we deduce that for every $\sigma > 0$, there is $\delta > 0$ such that

$\sup_y \int_{D_{i\delta}} |(Hy)(t)| dt < \frac{\sigma}{4.2^i}$,formeas$D_{i\delta} < \delta$, $D_{i\delta} \subset [i-1, i]$, $i = 1,2, \dots, n$.

Choose for each $i, i = 1,2, \dots, n, r_i^* \in N$ with $m([i-1,i]/D_{r_i^*}) < \delta$, then for $n_1, n_2 \in N$, we have

$\int_0^\infty \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt = \lim_{n\to\infty} \sum_{i=1}^n \int_{i-1}^i \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt$

$\leq \lim_{n\to\infty} \sum_{i=1}^n \int_{[i-1,i]/D_{r_i^*}} \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt + \int_{D_{r_i^*}} \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt$

$$\leq \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{2.\sigma}{4.2^i} + \left\|Hy_{n_1} - Hy_{n_2}\right\|_{c(D_{r_i^*})}\right)$$

$\leq \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{\sigma}{2.2^i} + \frac{\sigma}{2.2^i}\right) = \sum_{i=1}^\infty \frac{\sigma}{2^i} = \sigma$,

for large value of n_1, n_2 we deduce that $\{H(y_n)\}$ is Cauchy sequence in L_1, since L_1 is complete space, then $H(y_n)$ is relatively compact in L_1.

Finally, we can use Schauder fixed point theorem to get a fixed point for our operator H, so the functional integral equation (1.1) is solvable in L_1.

4. Example

Consider the integro-differential equation

$$x(t) = g(t) + \int_0^t k(t,s)\, f\left(s, x'(s)\right) ds \qquad (4.1)$$

We can transform this equation into another one, which is integral equation where, we differentiate both sides of equation (4.1) with respect to t .

Then, we have

$$x'(t) = g'(s) + k(t,t)\, f\left(t, x'(t)\right) + \int_0^t \frac{\partial k}{\partial t}(t,s)\, f\left(s, x'(s)\right) ds$$

Put $y(t) = x'(t)$, $g'(t) = h(t)$, $\frac{\partial k}{\partial t}(t,s) = p(t,s)$,$k(t,t) = q(t)$

Then we get

$$y(t) = q(t)\, f(t, y(t)) + h(t) + \int_0^t p(t,s)\, f\left(s, y(s)\right) ds \qquad (4.2)$$

Using our existence theorem, we deduce that the equation (4.2) and so (4.1) will be solvable under the assumptions of theorem (3.1).

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