Invariant Submanifolds of (k, μ)-Contact Manifold Admitting Quarter Symmetric Metric Connection

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Abstract— *In the present paper we have studied invariant submanifolds of a (k,μ)-contact manifold admitting quarter symmetric metric connection and obtained some interesting results.*

Keywords — *Invariant submanifold,(k,μ)-contact manifold, totally geodesic, quarter symmetric metric connection.*

I. INTRODUCTION

To study the geometry of an unknown manifold, it is sometime convenient and yet interesting to first imbed it into a rather known manifold and then study its geometry side by side that of the ambient manifold. This approach gave birth to the introduction of submanifold theory and which has become an independent research topic itself.

In 1995, Blair, Koufogiorgos and Papantoniou [5] introduced the notion of (k, μ)-contact manifold with an example, which generalizes the notion of Sasakian and the case $R(X,Y)\xi = 0$, where R is the curvature tensor, k, μ are real constants. While the study of invariant submanifolds of (k,μ)-contact manifold was initiated by Montano et al. [12] and followed by Tripathi et al. [16], Avik De [3], Siddesha and Bagewadi [14] and others.

On the other hand, we know that a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g on M such that $\nabla g = 0$, otherwise it is non-metric. Further, it is said to be semi-symmetric if its torsion tensor $T(X,Y) = 0$, i.e., $T(X,Y) = \omega(Y)X - \omega(X)Y = 0$, where ω is a 1-form. In 1924, Friedmann and Schouten [9] introduced the idea of semi-symmetric linear connection on differentiable manifold. In 1932, Hayden [11] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold was published by K. Yano [18]. In 1975, Golab [10] defined and studied quarter symmetric metric linear connection on a differentiable manifold. A linear connection $\bar{\nabla}$ in a n-dimensional Riemannian manifold is said to be a quarter symmetric connection [10] if its torsion tensor T is of the form

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = A(Y)KX - A(X)KY, \tag{1.1}$$

where A is a 1-form and K is a tensor field of type $(1,1)$. If a quarter symmetric metric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0,$$

for all $X,Y,Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, then $\bar{\nabla}$is said to be a quarter symmetric metric connection. For a contact metric manifold admitting quarter symmetric metric connection, we can take $A = \eta$ and $K = \phi$ to write $(1,1)$ in the form:

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{1.2}$$

The relation between Levi-Civita connection ∇ and quarter symmetric metric connection $\bar{\nabla}$ of a contact metric manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.3}$$

Recently in [1], the authors studied invariant submanifolds of Sasakian manifold admitting quarter symmetric metric connection and proved the equivalence of totally geodesicity, recurrence, bi-recurrence, generalized bi-recurrence of second fundamental form and a submanifold with parallel third fundamental form.

These circumstances motivated us to consider invariant submanifolds of (k,μ)-contact manifolds admitting quarter symmetric metric connection. The paper is organized as follows: In section 2, we give a brief information about recurrent manifolds and submanifolds. In section 3, some definitions and notions about (k,μ)-contact manifolds are given. Section 4 deals with the study of some basic results of invariant submanifolds of (k, μ)-contact manifolds admitting quarter symmetric metric connection. In section 5, we consider an invariant submanifolds of (k,μ)-contact manifold admitting quarter symmetric metric connection whose second fundamental form is recurrent, 2-recurrent and generalized 2-recurrent. We show that these type submanifolds are totally geodesic under necessary condition. We also prove that parallel submanifold and a submanifold with parallel third fundamental form of a (k,μ)-contact manifold admitting quarter symmetric metric connection with are again totally geodesic under necessary condition. In last section, we prove semi-parallel, pseudo-parallel and Ricci-generalized pseudo-parallel invariant submanifolds of a (k,μ)-contact manifold admitting quarter symmetric metric connection with $k \neq 0$ are totally geodesic.

II. IMMERSION OF RECURRENT TYPE

The covariant differential of the p^{th} order, $p \geq 1$, of a $(0,k)$-tensor field T, $k \geq 1$, defined on a Riemannian manifold (M,g) with the Levi-Civita connection ∇ is denoted by $\nabla^p T$. According to [13] the tensor T is said to be recurrent, respectively, 2- recurrent, if the following condition holds on M

$$(\nabla T)(X_1, \dots \dots, X_k; X)T(Y_1, \dots \dots, Y_k) = (\nabla T)(Y_1, \dots, Y_k X)T(X_1, \dots \dots, X_k) \qquad (2.1)$$

respectively

$$(\nabla^2 T)(X_1, \dots \dots, X_k; X, Y)T(Y_1, \dots \dots, Y_k) = (\nabla^2 T)(Y_1, \dots, Y_k X, Y)T(X_1, \dots \dots, X_k) \qquad (2.2)$$

where $X, Y, X_1, Y_1, \dots, X_k, Y_k \in TM$. From (2.1), it follows that at a point $x \in M$ if the tensor T is non-zero then there exists a unique 1-form $\emptyset$, respectively, a $(0,2)$-tensor ψ, defined on a neighbourhood U of x, such that

$$\nabla T = T \otimes \phi, \phi = d(\log \parallel T \parallel). \qquad (2.3)$$

respectively

$$\nabla^2 T = T \otimes \psi, \qquad (2.4)$$

holds on U, where $\parallel T \parallel$ denotes the norm of T, $\parallel T \parallel^2 = g(T,T)$. The tensor T is said to be generalized 2-recurrent if

$$((\nabla^2 T)(X_1, \dots \dots, X_k; X, Y) - (\nabla T \otimes \phi)(X_1, \dots \dots, X_k; X, Y))T(Y_1, \dots \dots, Y_k) = ((\nabla^2 T)(Y_1, \dots, Y_k X, Y) - (\nabla T \otimes \phi)(Y_1, \dots, Y_k X, Y)T(X_1, \dots \dots, X_k), \qquad (2.5)$$

holds on M, where ϕ is a 1 $-$form on M. From this it follows that at a point $x \in M$ if the tensor T is nonzero then there exists a unique $(0,2)$ $-$tensor ψ, defined on a neighbourhood U of x, such that

$$\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi \qquad (2.6)$$

holds on U.

Let $f : (M,g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion of an n-dimensional Riemannian manifold (M,g) into an $(n+d)$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$, $n \geq 2, d \geq 1$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}$, respectively. Then we have the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \qquad (2.7)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \qquad (2.8)$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero, then the manifold is said to be totally geodesic. The second fundamental form σ and A_N are related by

$$\tilde{g}(\sigma(X,Y), N) = g(A_N X, Y),$$

where g is the induced metric of $\tilde{g}$ for any vector fields X and Y tangent to M. The first and second covariant derivatives of the second fundamental form σ are given by

$$(\bar{\nabla}_X \sigma)(Y,Z) = \nabla_X^{\perp}(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \qquad (2.9)$$

$$(\bar{\nabla}^2 \sigma)(Z,W,X,Y) = (\bar{\nabla}_X \bar{\nabla}_Y \sigma)(Z,W) = \nabla_X^{\perp}((\bar{\nabla}_Y \sigma)(Z,W)) - (\bar{\nabla}_Y \sigma)(\nabla_X Z, W) - (\bar{\nabla}_X \sigma)(Z, \nabla_Y W) - (\bar{\nabla}_{\nabla_X Y} \sigma)(Z,W), \qquad (2.10)$$

respectively, where $\bar{\nabla}$ is called the vander Waerden-Bortolotti connection of M [9]. If $\bar{\nabla}\sigma = 0$, then M is said to have parallel second fundamental form [9]. Now for a $(0,k)-$ tensor $T, k \leq 1$ and a $(0,2)$ $-$tensor B, $Q(B,T)$ is defined by [16]

$$Q(B,T)(X_1, X_2, \dots, X_k; X, Y) = -T((X \wedge_B Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, (X \wedge_B Y)X_2, \dots, X_k), \qquad (2.11)$$

where $(X \wedge_B Y)$ is defined as

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y. \qquad (2.12)$$

III. (k,μ) $-$CONTACT MANIFOLDS

A contact manifold is a $C^{\infty} - (2n+1)$ manifold $\tilde{M}^{2n+1}$ equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $\tilde{M}^{2n+1}$. Given a contact form η it is well known that there exists a unique vector field ξ, called the characteristic vector field of η, such that $\eta(\xi) = 1$ and $d\eta(X,\xi) = 0$ for every vector field X on $\tilde{M}^{2n+1}$. A Riemannian metric is said to be associated metric if there exists a tensor field φ of type $(1,1)$ such that

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \cdot \phi = 0 \qquad (3.1)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X) \qquad (3.2)$$

for all vector fields $X, Y \in T\tilde{M}$. Then the structure (φ, ξ, η, g) on $\tilde{M}^{2n+1}$ is called a contact metric structure and the manifold $\tilde{M}^{2n+1}$ equipped with such a structure is called a contact metric manifold [4]. Now for any vector fields X, Y on $\tilde{M}$, a contact metric manifold is called a (k,μ)-contact manifold [5] if it satisfies

$$(\tilde{\nabla}_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \qquad (3.3)$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$, k, μ are real constants and $2h$ is the Lie derivative of φ in the direction of ξ. We also have on (k,μ) −contact manifold $\tilde{M}$

$$\tilde{\nabla}_X \xi = -\varphi X - \varphi hX. \qquad (3.4)$$

for any vector fields X, Y, Z, the following relations are holds in (k,μ)-contact manifold [12]:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \qquad (3.5)$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y) - \eta(Y)(X + hX), \qquad (3.6)$$

$$R(\xi, X)\xi = k[\eta(X)\xi - X] - \mu hX \qquad (3.7)$$

$$S(X,\xi) = 2nk\eta(X), \qquad (3.8)$$

$$r = 2n(2n - 2 + k - n\mu), \qquad (3.9)$$

where R is the Riemannian curvature tensor, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

IV. Some basic results on invariant submanifolds of (k, μ)-contact manifold admitting quarter symmetric metric connection

Let M be a submanifold of a (k,μ)-contact manifold $\tilde{M}$, then M is called an invariant submanifold of $\tilde{M}$, if for each $x \in M$, $\phi(T_x M) \subset T_x M$. As a consequence, ξ becomes tangent to M. For an invariant submanifold of a (k,μ)-contact manifold, we have

$$\sigma(X,\xi) = 0, \qquad (4.1)$$

for any $X \in TM$.

In [1], the authors have studied invariant submanifolds of Kenmotsu manifold admitting quarter symmetric metric connection and obtained several results. To prove the main results, we mention some basic results of [1] regarding to quarter symmetric metric connection:

Lemma 4.1. *Let M be an invariant submanifold of a contact manifold $\tilde{M}$ which admits quarter symmetric metric connection $\bar{\tilde{\nabla}}$ and let σ and $\bar{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and quarter symmetric metric connection respectively, then*

(i) M admits quarter symmetric metric connection,

(ii) the second fundamental forms with respect to $\tilde{\nabla}$ and $\bar{\tilde{\nabla}}$ are equal.

Lemma 4.2. *Let M be an invariant submanifold of a contact manifold $\tilde{M}$ which admits quarter symmetric metric connection. Then Gauss and Weingarten formulas with respect to quarter symmetric metric connection are given by*

$$\begin{aligned}
tan\left(\bar{\tilde{R}}(X,Y)Z\right) &= R(X,Y)Z - \eta(X)\phi\nabla_Y Z \\
&-\eta(Y)\nabla_X \phi Z + \eta(Y)\phi\nabla_X Z + \eta(X)\nabla_Y \phi Z \\
&+\eta[X,Y]\phi Z + \tan\{\bar{\tilde{\nabla}}_X \sigma(Y,Z) - \bar{\tilde{\nabla}}_Y \sigma(X,Z) \\
&+\bar{\tilde{\nabla}}_Y \eta(X)\phi Z - \bar{\tilde{\nabla}}_X \eta(Y)\phi Z\},
\end{aligned} \qquad (4.2)$$

$$\begin{aligned}
nor\left(\bar{\tilde{R}}(X,Y)Z\right) &= \sigma(X,\nabla_Y Z) - \eta(Y)\sigma(X,\phi Z) \\
&-\sigma(Y,\nabla_X Z) + \eta(X)\sigma(Y,\phi Z) - \sigma([X,Y],Z) \\
&+nor\{\bar{\tilde{\nabla}}_X \sigma(Y,Z) - \bar{\tilde{\nabla}}_Y \sigma(X,Z) + \bar{\tilde{\nabla}}_Y \eta(X)\phi Z \\
&-\bar{\tilde{\nabla}}_X \eta(Y)\phi Z\},
\end{aligned} \qquad (4.3)$$

Lemma 4.3. *Let M be an invariant submanifold of a contact manifold $\tilde{M}$ which admits quarter symmetric metric connection. Then*

$$\begin{aligned}
(\bar{\tilde{R}}(X,Y)\cdot\sigma(U,V) &= R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) \\
&-\sigma(U,R(X,Y)V) - \nabla_X A_{\sigma(U,V)}Y - \sigma\left(X, A_{\sigma(U,V)}Y\right) \\
&+\eta(X)\phi A_{\sigma(U,V)}Y - A_{\nabla^{\perp}Y\sigma(U,V)}X - \eta(X)\phi\nabla^{\perp}Y\sigma(U,V) \\
&-\bar{\tilde{\nabla}}_X \eta(Y)\phi\sigma(U,V) + \nabla_Y A_{\sigma(U,V)}X + \sigma\left(Y, A_{\sigma(U,V)}X\right) \\
&+\bar{\tilde{\nabla}}_Y \eta(X)\phi\sigma(U,V) + A_{\sigma(U,V)}[X,Y] + \eta([X,Y]) \\
&\phi\sigma(U,V) - \sigma(\sigma(X,\nabla_Y U),V) + \eta(X)\sigma(\phi\nabla_Y U,V) \\
&-\sigma(\bar{\tilde{\nabla}}_X\{\sigma(Y,U),V) + \sigma\left(\bar{\tilde{\nabla}}\eta(Y)\phi U,V\right) \\
&+\eta(Y)\sigma(\nabla_X \phi U,V) + \eta(Y)\sigma(\sigma(X,\phi U),V) \\
&+\sigma(\sigma(Y,\nabla_X U),V) - \eta(Y)\sigma(\phi\nabla_X U,V) \\
&+\sigma\left(\bar{\tilde{\nabla}}_Y\{\sigma(X,U)\},V\right) - \sigma\left(\bar{\tilde{\nabla}}_Y\eta(X)\phi U,V\right) \\
&-\eta(X)\sigma(\nabla_Y \phi U,V) - \eta(X)\sigma(\sigma(Y,\phi U),V) \\
&+\sigma(\sigma([X,Y],U),V) - \eta([X,Y])\sigma(\phi U,V) \\
&-\sigma\left(U,\sigma(X,\nabla_Y V)\right) + \eta(X)\sigma(U,\phi\nabla_Y V) \\
&-\sigma\left(U,\bar{\tilde{\nabla}}_X\{\sigma(Y,V)\}\right) + \sigma\left(U,\bar{\tilde{\nabla}}_X\eta(Y)\phi V\right) \\
&+\eta(Y)\sigma(U,\nabla_X \phi V) + \eta(Y)\sigma\left(U,\sigma(X,\phi V)\right) \\
&+\sigma\left(U,\sigma(Y,\nabla_X V)\right) - \eta(Y)\sigma(U,\phi\nabla_X V) \\
&+\sigma\left(U,\bar{\tilde{\nabla}}_Y\{\sigma(X,V)\}\right) - \sigma\left(U,\bar{\tilde{\nabla}}_Y\eta(X)\phi V\right) \\
&-\eta(X)\sigma(U,\nabla_Y \phi V) - \eta(X)\sigma\left(Y,\sigma(U,\phi V)\right) \\
&+\sigma(U,\sigma([X,Y]),V) - \eta([X,Y])\sigma(U,\phi V),
\end{aligned} \qquad (4.4)$$

for all vector fields X, Y, U and V tangent to M, where $R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}$.

V. RECURRENT INVARIANT SUBMANIFOLDS OF (k, μ)- CONTACT MANIFOLD ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

In this section, we consider invariant submanifolds of a (k,μ)-contact manifold when σ is recurrent, 2-recurrent, generalized 2-recurrent and M has parallel third fundamental form with respect to quarter symmetric metric connection. We write the equations (2.9) and (2.10) with respect to semi-symmetric metric connection in the form

$$\begin{aligned}
(\bar{\tilde{\nabla}}_X \sigma(X,Y) &= \bar{\nabla}^{\perp}_X\left(\sigma(Y,Z)\right) - \sigma\left(\bar{\nabla}Y,Z\right) \\
&-\sigma(Y,\bar{\nabla}_X Y)
\end{aligned} \qquad (5.1)$$

$$\begin{aligned}
\bar{\tilde{\nabla}}^2\sigma(Z,W,X,Y) \\
&= \left(\bar{\tilde{\nabla}}_X\bar{\tilde{\nabla}}_Y\right)(Z,W) \\
&= (\bar{\nabla}^{\perp}_X\left(\bar{\tilde{\nabla}}_Y\sigma\right)(Z,W))\text{-}(\bar{\tilde{\nabla}}_Y\sigma)(\bar{\nabla}_X Z,W)
\end{aligned}$$

$$-(\bar{\nabla}_X\sigma)(Z,\bar{\nabla}_{\bar{\nabla}_XY}\sigma)(Z,W). \quad (5.2)$$

We prove the following theorems:

Theorem 5.1: *Let M be an invariant submanifold of (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then M is parallel with respect to quarter symmetric metric connection with $k \neq 0$ if and only if M is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M be parallel with respect to quarter symmetric metric connection. Then we have

$$(\bar{\nabla}\sigma)(Y,Z)=0.$$

By taking $Z=\xi$ and applying (5.1) in the above equation, we get

$$\bar{\nabla}_X^{\perp}\sigma(Y,\xi)-\sigma(\bar{\nabla}_XY,\xi)-\sigma(Y,\bar{\nabla}_X\xi)=0.$$

In view of (4.1), the above equation reduces to

$$-\sigma(\bar{\nabla}_XY,\xi)-\sigma(Y,\bar{\nabla}_X\xi)=0. \quad (5.3)$$

Using (1.2), (3.4) and (4.1) in (5.3), we get

$$\sigma(Y,\phi X+\phi hX)=0. \quad (5.4)$$

Replace X by $\phi X+\phi hX$ and by virtue of (3.1), (3.5), (4.1) in (5.4), we get

$$k\sigma(Y,X)=0.$$

Since $k\neq 0$, hence M is totally geodesic with respect to Levi-Civita connection. The converse statement is trivial. This completes the proof of the theorem.

Corollary 5.1: *Let M be an invariant submanifold of (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then σ is recurrent with respect to quarter symmetric metric connection with $k\neq 0$ if and only if M is totally geodesic with respect to Levi-Civita connection.*

Proof. Since σ is recurrent with respect to semi-symmetric metric connection. Then from (2.3), we have

$$\bar{\nabla}_X\sigma(Y,Z)=\phi(X)\sigma(Y,Z),$$

where ϕ is a 1-form. Taking $Z=\xi$ in the above equation and using the proof of the ***Theorem 5.1***, we get the result.

Theorem 5.2. *Let M be an invariant submanifold of (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then M has parallel third fundamental form with respect to quarter symmetric metric connection with $k\neq 0$ if and only if M is totally geodesic with respect to Levi-Civita connection.*

Proof: Suppose that M has parallel third fundamental form with respect to semi-symmetric metric connection. Then we can write

$$(\bar{\nabla}_X\bar{\nabla}_Y\sigma)(Z,W)=0.$$

Replacing $W=\xi$ in the above equation and using (5.2), we obtain

$$\bar{\nabla}_X^{\perp}\left((\bar{\nabla}_Y\sigma)(Z,\xi)\right)-(\bar{\nabla}_Y\sigma)(\bar{\nabla}_XZ,\xi)$$
$$-(\bar{\nabla}_X\sigma)(Z,\bar{\nabla}_Y\xi)-(\bar{\nabla}_{\bar{\nabla}_XY}\sigma)(Z,\xi)=0. \quad (5.5)$$

By using (4.1) and (5.1) in (5.5), we obtain

$$\begin{aligned}0=&-\bar{\nabla}_X^{\perp}\{\sigma(\bar{\nabla}_YZ,\xi)+\sigma(Z,\bar{\nabla}_Y\xi)\}\\&-\bar{\nabla}_Y^{\perp}\sigma(\bar{\nabla}_YZ,\xi)+\sigma(\bar{\nabla}_Y\bar{\nabla}_XZ,\xi)\\&+2\sigma(\bar{\nabla}_XZ,\bar{\nabla}_Y\xi)-\bar{\nabla}_X^{\perp}\sigma(Z,\bar{\nabla}_Y\xi)\\&-\sigma(Z,\bar{\nabla}_X\bar{\nabla}_Y\xi)+\sigma(\bar{\nabla}_{\bar{\nabla}_XY}Z,\xi)\\&+\sigma(Z,\bar{\nabla}_{\bar{\nabla}_XY}\xi).\end{aligned} \quad (5.6)$$

In view of (1.2), (3.4) and (4.1) in (5.6) yields

$$\begin{aligned}0=&\,2\bar{\nabla}_X^{\perp}\sigma(Z,\phi Y+\phi hY)-2\sigma(\nabla_XZ,\phi Y+\phi hY)\\&+2\eta(X)\sigma(\phi Z,\phi Y+\phi hY)+\sigma(Z,\nabla_X(\phi Y+\phi hY))\\&-\sigma(Z,\phi\nabla_XY+\phi h\nabla_XY)+\eta(X)\sigma(Z,Y+hY).\end{aligned}$$

Taking $Z=\xi$ and using (4.1), (3.1) and (3.4) in the above equation, we obtain

$$2\sigma(\phi X+\phi hX,\ \phi Y+\phi hY)=0. \quad (5.7)$$

Replacing X by $\phi X+\phi hX$, Y by $\phi Y+\phi hY$ and using (3.4), (3.5) in (5.7), one can obtain

$$2k^2\sigma(X,Y)=0. \quad (5.8)$$

Since $k\neq 0$, hence M is totally geodesic with respect to Levi-Civita connection. The converse statement is trivial. This completes the proof of the theorem.

Corollary 5.1: *Let M be an invariant submanifold of (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then σ is 2-recurrent with respect to quarter symmetric metric connection with $k\neq 0$ if and only if M is totally geodesic with respect to Levi-Civita connection.*

Proof. Since σ is 2-recurrent with respect to semi-symmetric metric connection. Then from (2.3), we have

$$(\bar{\nabla}_X\bar{\nabla}_Y\sigma)(Z,W)=\sigma(Z,W)\phi(X,Y). \quad (5.9)$$

Taking $Z=\xi$ in (5.9) and using the proof of the **Theorem 5.2**, we get the result.

Theorem 5.3. *Let M be an invariant submanifold of (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then σ is generalized 2-recurrent with respect to quarter symmetric metric connection with $k\neq 0$ if and only if M is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be generalized 2-recurrent with respect to quarter symmetric metric connection. From (2.6), we can write

$$(\bar{\nabla}_X\bar{\nabla}_Y\sigma)(Z,W)=$$
$$\psi(X)\sigma(Z,W)+\phi(X)(\bar{\nabla}_Y\sigma)(Z,W)$$

,

where ψ and ϕ are 2-form and 1-form respectively. Taking $W = \xi$ in the above equation and using (4.1) and (5.2), we have

$$\overline{\nabla}_X^{\perp}\left((\overline{\nabla}_Y\sigma)(Z,\xi)\right) - (\overline{\nabla}_Y\sigma)(\overline{\nabla}_X Z,\xi)$$
$$-(\overline{\nabla}_X\sigma)(Z,\overline{\nabla}_Y\xi) - (\overline{\nabla}_{\overline{\nabla}_X Y}\sigma)(Z,\xi)$$
$$= -\phi(X)\{\sigma(\overline{\nabla}_Y Z,\xi) + \sigma(Z,\overline{\nabla}_Y\xi)\}. \qquad (5.13)$$

Then making use of (5.1) and in view of (4.1), we have

$$-\overline{\nabla}_X^{\perp}\{\sigma(\overline{\nabla}_Y Z,\xi) + \sigma(Z,\overline{\nabla}_Y\xi)\} - \overline{\nabla}_Y^{\perp}\sigma(\overline{\nabla}_Y Z,\xi)$$
$$+\sigma(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi) + 2\sigma(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi) - \overline{\nabla}_X^{\perp}\sigma(Z,\overline{\nabla}_Y\xi)$$
$$-\sigma(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi) + \sigma(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi) + \sigma(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi)$$
$$= -\phi(X)\{\sigma(\overline{\nabla}_Y Z,\xi) + \sigma(Z,\overline{\nabla}_Y\xi)\} \qquad (5.14)$$

Taking $Z = \xi$ and using (3.1), (3.4), (3.5) and (4.1) in (5.12), we get equation (5.8). Hence proceeding in a similar way of proof of ***Theorem 5.2.***, we get $\sigma(X,Y) = 0$.

VI. *SEMI-PARALLEL, PSEUDO-PARALLEL, RICCI-GENERALIZED PSEUDO-PARALLEL INVARIANT SUBMANIFOLDS OF (k, μ)-CONTACT MANIFOLDS ADMITTING QUARTER SYMMETRIC METRIC*

This section deals with semi-parallel, pseudo-parallel and Ricci-generalized pseudo-parallel invariant submanifolds of (k, μ)-contact manifold admitting quarter symmetric metric connection

Definition 6.1. An immersion is said to be semi-parallel, pseudo-parallel and Ricci-generalized pseudo-parallel [7, 8] with respect to quarter symmetric metric connection $\tilde{\overline{\nabla}}$, respectively, if

$$\tilde{\overline{R}}\cdot\sigma = 0,$$
$$\tilde{\overline{R}}\cdot\sigma = L_1 Q(g,\sigma),$$
$$\tilde{\overline{R}}\cdot\sigma = L_2 Q(S,\sigma),$$

where $\tilde{\overline{R}}$ denotes the curvature tensor with respect to connection $\tilde{\overline{\nabla}}$ and L_1, L_2 are the functions depending on σ.

To prove the main theorems, we first formulate the following lemma:

Lemma 6.4. *[14] It is known that if (M,ϕ,ξ,η,g) is a contact Riemannian manifold and ξ belongs to the (k,μ)-nullity distribution, then $k \le 1$, if $k < 1$, then (M,ϕ,ξ,η,g) admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ defined by the eigen spaces of h, where $\lambda = \sqrt{1-k}$. Further, if $X \in D(\lambda)$, then $hX = \lambda X$ and if $X \in D(-\lambda)$, then $hX = -\lambda X$.*

Theorem 6.4. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then M is semi-parallel with respect to quarter symmetric metric connection with $k \neq \pm[(\mu+1)\sqrt{1-k}] - 1$ if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof: Suppose M is semi-parallel with respect to quarter symmetric metric connection, then we have

$$\tilde{\overline{R}}\cdot\sigma = 0.$$

Taking $X = V = \xi$ and using (3.1), (3.4), (4.1) and (4.4) to get

$$0 = -\sigma(U,R(\xi,Y)\xi) - \sigma(\overline{\nabla}_\xi\sigma(Y,U),\xi)$$
$$+\sigma(\overline{\nabla}_\xi\eta(Y)\phi U,\xi) - \sigma(\overline{\nabla}_\xi\phi U,\xi)$$
$$+\sigma(U,\phi\nabla_Y\xi). \qquad (6.1)$$

By definition σ is a vector valued covariant tensor and so $\sigma(U,Y)$ is a vector. Therefore $\overline{\nabla}_\xi\sigma(Y,U)$ is a vector and hence by (4.1), we have

$$\sigma(\overline{\nabla}_\xi\sigma(Y,U),\xi) = 0. \qquad (6.2)$$

Using (1.2), (2.1), (3.1), (3.4), (3.7), (4.1) and (6.2) in (6.1), we get

$$(k+1)\sigma(U,Y) + (\mu+1)\sigma(U,hY) = 0. \qquad (6.3)$$

Therefore Lemma 6.4 yields $[(k+1) \pm (\mu+1)]\sigma(U,Y) = 0$. Which implies that $\sigma(U,Y) = 0$ provided $k \neq \pm[(\mu+1)\sqrt{1-k}] - 1$. Thus M is totally geodesic with respect to quarter symmetric metric connection $k \neq \pm[(\mu+1)\sqrt{1-k}] - 1$. The converse statement is trivial. This completes the proof of the theorem.

Theorem 6.5. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then M is pseudo-parallel with respect to quarter symmetric metric connection with $L_1 \neq [k+1 \pm (\mu+1)\sqrt{1-k}]$ if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof: Suppose M is pseudo-parallel with respect to quarter symmetric metric connection, then we have

$$\tilde{\overline{R}}\cdot\sigma = L_1 Q(g,\sigma).$$

Taking $X = V = \xi$ and use (3.1), (3.4) and (4.1) in (2.11) and (4.4) to get

$$-\sigma(U,R(\xi,Y)\xi) - \sigma(\overline{\nabla}_\xi\sigma(Y,U),\xi)$$
$$+\sigma(\overline{\nabla}_\xi\eta(Y)\phi U,\xi) - \sigma(\overline{\nabla}_\xi\phi U,\xi)$$
$$+\sigma(U,\phi\nabla_Y\xi) = L_1\sigma(Y,U). \qquad (6.4)$$

Using (1.2), (2.7), (3.1), (3.4), (3.7), (4.1) and (6.2) in (6.4), we obtain

$$(k+1)\sigma(U,Y) + (\mu+1)\sigma(U,hY)$$
$$= L_1\sigma(Y,U). \qquad (6.5)$$

In view of Lemma 6.4, equation (6.5) can be written as

$$(L_1 - [(k+1) \pm (\mu+1)\sqrt{1-k}])\sigma(U,Y) = 0$$

Hence $\sigma(U,Y)=0$ provided $L_1 \neq [k+1 \pm (\mu+1)\sqrt{1-k}]$.

Thus M is totally geodesic with respect to quarter symmetric metric connection provided $L_1 \neq [k+1 \pm (\mu+1)\sqrt{1-k}]$. The converse statement is trivial. This completes the proof of the theorem.

Theorem 6.6. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then M is Ricci-generalized pseudo-parallel with respect to quarter symmetric metric connection with $L_2 \neq \frac{[k+1 \pm (\mu+1)\sqrt{1-k}]}{2nk}$ if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof: Let M be Ricci-generalized pseudo-parallel with respect to quarter symmetric metric connection, then by definition 6.1. we have

$$\bar{\tilde{R}} \cdot \sigma = L_2 Q(S,\sigma).$$

Taking $X = V = \xi$ and use (3.1), (3.4), (4.1) in (2.11) and (4.4) to get

$$\begin{aligned} &-\sigma(U,R(\xi,Y)\xi) - \sigma(\bar{\nabla}_\xi \sigma(Y,U),\xi) \\ &+\sigma(\bar{\nabla}_\xi \eta(Y)\phi U,\xi) - \sigma(\bar{\nabla}_\xi \phi U,\xi) \\ &+\sigma(U,\phi\nabla_Y \xi) = L_2 \sigma(S(\xi,\xi)Y,U). \end{aligned} \qquad (6.6)$$

Using (1.2), (2.7), (3.1), (3.4), (3.7), (3.8), (4.1) and (6.2) in (6.6), we obtain

$$\begin{aligned} (k+1)\sigma(U,Y) + (\mu+1)\sigma(U,hY) \\ = 2nkL_2\sigma(Y,U). \end{aligned} \qquad (6.7)$$

In view of Lemma 6.4, equation (6.7) can be written as

$$(2nkL_2 - [(k+1) \pm (\mu+1)\sqrt{1-k}])\,\sigma(U,Y) = 0.$$

Hence $\sigma(U,Y)=0$ provided $L_2 \neq \frac{[k+1 \pm (\mu+1)\sqrt{1-k}]}{2nk}$.

Thus M is totally geodesic with respect to quarter symmetric metric connection provided $L_2 \neq \frac{[k+1 \pm (\mu+1)\sqrt{1-k}]}{2nk}$. The converse statement is trivial. This completes the proof of the theorem.

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