Hamilton Decomposition of Harary Graphs

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Abstract — *Hamilton decomposition is one of the earliest concepts in the field of graph theory. The Hamilton decomposability problem plays a vital role in the study of Combinatorial Design theory. A Harary graph $H_{k,n}$ is a k-connected simple graph with n vertices and with a minimal number of edges. Constructing Harary graphs and learning about their connectivity is considered to be one of those typical mathematical topics. In fact, the connectivity of Harary graphs addresses a very relevant question in communication networks: trading off the costs between reliability and the number of communication links. In this paper, we investigate the Hamilton decomposition of Harary graphs and its minimum bound for Hamilton decomposition.*

Keywords — *Hamilton decomposition, Harary graphs.*

I. INTRODUCTION

Decomposition of graphs is one of the prominent areas of research in graph theory and combinatorial design theory. The purpose of the study of decomposition of a graph G is to find that number n for which the graph G can be distorted into n copies of a fixed graph or n different graphs. The significance of decomposition is that they have vital applications in various fields of study such as networking, block designing and bioinformatics.

Decomposition of a graph $G(V,E)$ is the set of all distinct edge-disjoint subgraphs $H_1, H_2,...,H_r$ of G such that their union is G itself where every edge of G belongs to exactly one H_i [4].

Various types of decompositions have been cited by different authors. Decomposition of a graph G is called *path decomposition* if all the $H_i s$ are paths. Decomposition of a graph G is called *cycle decomposition* if all the $H_i s$ are cycles. Arumugam S et al. studied on decomposition of graphs into paths and cycles in general [4] while Dolober Froneck studied on the decomposition of complete graphs into small graphs [5].

Hamilton nature of a graph is by itself a vast area of study in graph theory, so Hamilton decomposition has been widely studied and applied in many fields. A vast study on the necessary and sufficient condition for a graph to be Hamilton decomposable can be cited in literature.

The most acceptable necessary and sufficient condition for a graph to be Hamilton decomposable was given by C. St. J. A. Nash-William's [10]

conjecture "Every *2k* - regular graph with atmost *4k+1* vertices has a Hamilton decomposition". This statement is accepted but does not have a satisfactory proof. A graph G is said to be *Hamilton decomposable* if all it's edge-disjoint subgraphs are spanning cycles.

The topic under discussion is the partition of the edge set of a graph into Hamilton cycles. An obvious necessary condition for such a partition to exist is that the graph has to be regular of even degree. However, if a graph is of odd degree, then the closest one can come is to partition the edge set into a single perfect matching and Hamilton cycles. Hence, we redefine the Hamilton decomposition of a graph G in terms of its regularity as follows:

Let G be a regular graph with edge set $E(G)$. It is said to have Hamilton decomposition or said to be Hamilton decomposable if either *deg(G)=2d* and *E(G)* can be partitioned into *d* Hamilton cycles or *deg(G)=2d+1* and *E(G)* can be partitioned into *d* Hamilton cycles and a perfect matching.

The earliest results about Hamilton decomposition deal with the complete graph K_n [8] where in it is studied that complete graph K_n has edge-disjoint cycle cover for every *n ≥ 3*. Depending on the connectivity and regularity of the network, Hamilton decomposition of many other networks was studied. Albert Williams et al. have studied the Hamilton decomposition of Butterfly and Benes network [2].

The first result in this direction is Walecki's construction of Hamilton decomposition for the complete graph on an odd number of vertices. In 1967, C. St. J. A. Nash Williams [11] gave the necessary and sufficient condition for a graph to be Hamilton decomposable. Tillson [15] proved an analogue of this for complete digraphs.

In 1968, Kelly conjectured an analogue of this for tournaments, namely that every regular tournament has a Hamilton decomposition. Note that in a digraph we allow upto two edges between any pair of vertices – atmost one in each direction. So, a digraph might contain cycles of length two, where as in an oriented graph, only atmost one edge between any pair of vertices is allowed. A tournament T is an orientation of a complete (undirected) graph. T is regular if the out degree of every vertex equals its indegree. In this paper we study decomposition concepts for undirected simple graphs.

It is necessary that the decomposition of any network into sub graphs has a minimum bound [1] as it validates the necessity of decomposition. Also

decomposition of any graph depends on the maximum connectivity of a graph [7] as connectivity plays a important role in decomposing a graph into edge disjoint subgraphs.

II. HAMILTON DECOMPOSITION OF HARARY GRAPHS

In this paper, we discuss the Hamilton decomposition of Harary graphs and its minimum bound of Hamilton decomposition. A communication network constructed as a Harary graph $H_{k,n}$ tells us that we can remove up to k vertices before the network becomes partitioned or in other words the network is split into components. This means that if we are considering networks that are designed to disseminate data to every node, Harary graphs will give us the means to make them just as robust as we want them to be, yet with a minimal number of links [10].

A Harary graph $H_{k,n}$ [10] is a k-connected simple graph with n vertices and with a minimal number of edges. $H_{k,n}$ has exactly $\lceil k.n/2 \rceil$ edges, that is, the smallest natural number of edges greater or equal to $k.n/2$. To this end, we label the vertices in $H_{k,n}$ as $\{0,1,2,...,n-1\}$ and organize them graphically as a circle. Following Bondy and Murty , we consider the following three cases for combinations of k and n:

Case (i): k is even
We construct $H_{k,n}$ by joining each vertex i to its $k/2$ closest left-hand (i.e., clockwise) neighbours and its $k/2$ closest right-hand (i.e., counter clockwise) neighbours.

Case (ii): k is odd, n is even
In this case, we construct $H_{k-1,n}$ and add $n/2$ edges by joining vertex i to its left-hand neighbour at distance $n/2$ (with $0 \le i \le n/2$). In other words, we add edges
$$\left(0,\frac{n}{2}\right),\left(1,1+\frac{n}{2}\right),\left(2,2+\frac{n}{2}\right),...,\left(\frac{n-2}{2},n-1\right).$$

Case (iii): k is odd, n is odd
In this case, we again first construct $H_{k-1,n}$ and then add the $(n+1)/2$ edges $\left(0,\frac{n-1}{2}\right)$, $\left(1,1+\frac{n-1}{2}\right)$, $\left(2,2+\frac{n-1}{2}\right),...,\left(\frac{n-1}{2},n-1\right).$

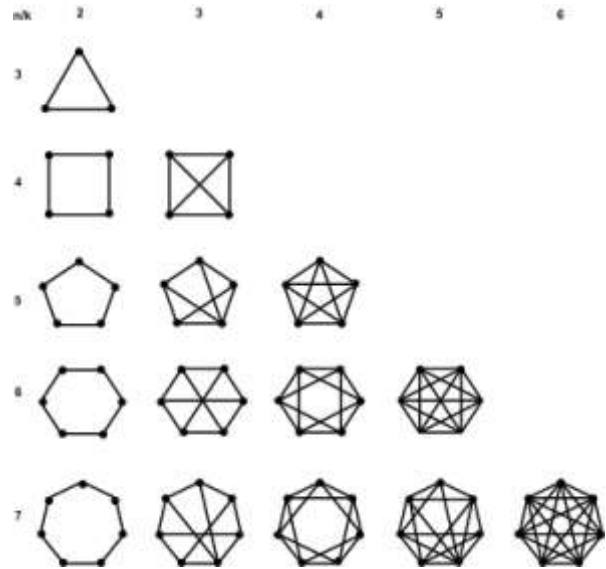An array of Harary graphs is given as follows:

Fig. 1 An Array of Harary graphs, $H_{k,n}$

We give a procedure to decompose Harary graphs into edge-disjoint Hamilton cycles. The possibility of this decomposition is due to the fact that $H_{k,n}$ is k-connected by definition and the fact that it is *2d-regular* when k is even and *(2d+1)*-regular (except for an irregularity at exactly one vertex). It is justified from the definition of Hamilton decomposition of graphs in accordance with their regularity mentioned in Section I.

In the process of generating the Hamilton decomposition of Harary graphs $H_{k,n}$, we investigated that $H_{k,n}$ can be decomposed into $k/2$ cycles each of length n when k is even and $\lfloor k/2 \rfloor$ cycles and $\lfloor n/2 \rfloor$ paths each of length n and 1 respectively. We now justify this argument.

III. PROCEDURE FOR HAMILTON DECOMPOSITION OF HARARY GRAPHS

We investigate the Hamilton decomposition of Harary graphs in accordance with the parity of   as follows:

A. Procedure for k/2- Hamilton Decomposition of $H_{k,n}$, for all $k \ge 2$, k is Even.

Consider the Harary graph $H_{k,n}$, where k is even. It's vertex set is given by $V = \{v_1,v_2,...,v_{i-1},v_i,v_{i+1},...,v_n\}$ where each v_i is adjacent to $k/2$ closest neighbours on the right and $k/2$ closest neighbours on the left.

Construct the edge-disjoint Hamilton cycles as follows:

Step 1: Consider the vertex v_1. Then by the construction of Harary graphs v_1 is adjacent to neighbours $v_2, v_3, ..., v_{1+k/2}$ on the right and neighbours $v_n, v_{n-1}, ..., v_{n-\frac{k}{2}+1}$ on the left.

Step 2: Consider the origin and terminus of every edge-disjoint Hamilton cycle of $H_{k,n}$ as v_1.

Step 3: Define the starting and terminal edges of such edge-disjoint Hamilton cycles to be

$$\left\{ v_1 v_2, v_1 v_3, \dots, v_1 v_{1+\frac{k}{2}} \right\}$$

and

$$\left\{ v_1 v_n, v_1 v_{n-1}, \dots, v_1 v_{n-\frac{k}{2}+1} \right\}$$

respectively such that the order is maintained and distinct. Hence, the origin and terminal edge of every $k/2$ Hamilton cycle of $H_{k,n}$, k is even is defined.

Step 4: To define the internal vertices of these edge-disjoint Hamilton cycles obtain the permutation σ for the vertices

$$\left\{ v_{1+\frac{k}{2}}, v_{2+\frac{k}{2}}, \dots, v_{n-\frac{k}{2}-1}, v_{n-\frac{k}{2}} \right\}.$$

Step 5: To complete the edge-disjoint Hamilton cycles we consider only those permutations from σ in which for any pair of indices $ij, 1 + \frac{k}{2} \le i, j \le n - \frac{k}{2}$, the edge $v_i v_j$ occurs exactly once in all the $k/2$ edge-disjoint Hamilton cycles of $H_{k,n}$, k is even.
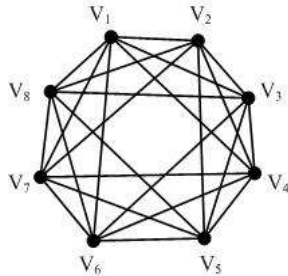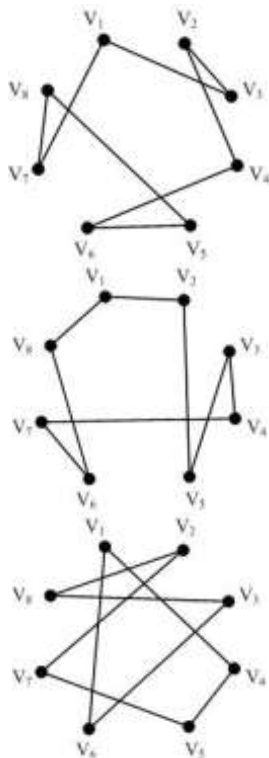

Fig. 2 The Harary Graph $H_{6,8}$

Fig. 3 Hamilton decomposition of $H_{6,8}$

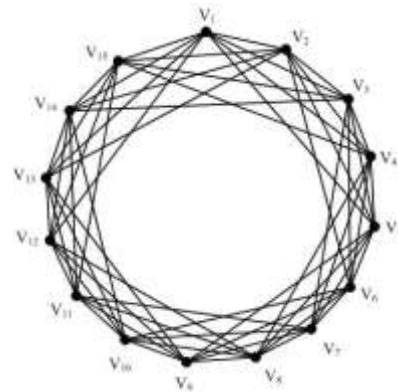$H_{6,8}$ is decomposed into *3* cycles, each of length *8*.
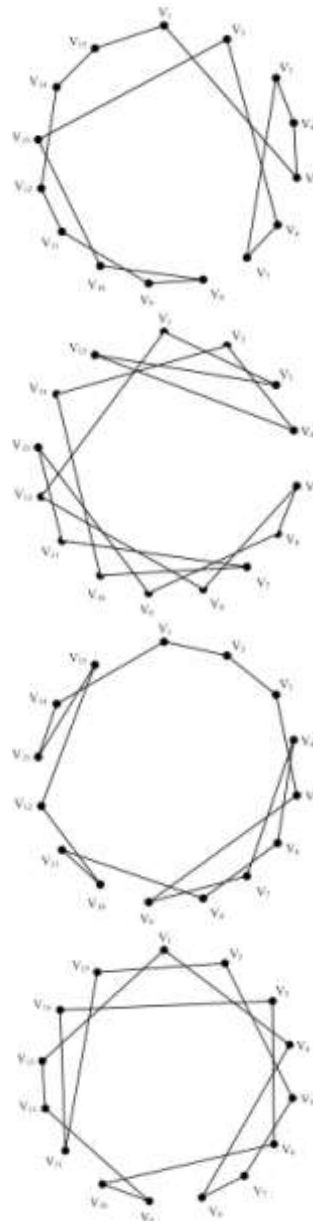

Fig. 4 The Harary Graph $H_{8,15}$

Fig. 5 Hamilton decomposition of $H_{8,15}$

$H_{8,15}$ is decomposed into *4* cycles, each of length *15*.

B. Procedure for Hamilton Decomposition of $H_{k,n}$, k is Odd.

By the construction of Harary graph we see that for an odd k, $H_{k,n}$ is constructed by joining $n/2$ edges when n is even and $1 + \left\lfloor \dfrac{n}{2} \right\rfloor$ edges when n is odd to $H_{k-1,n}$.

Hence, by the previous procedure $H_{k-1,n}$ can be decomposed into $(k-1)/2$ cycles, since, $k-1$ is even when k is odd. Also, it can be seen that by the construction of $H_{k,n}$, when k is odd the edges excluding those from $H_{k-1,n}$ are paths of length *1* (when n is odd there is exactly one path of length *2*). Therefore, when n is even, $n/2$ paths are obtained and when n is odd $1 + \left\lfloor \dfrac{n}{2} \right\rfloor$ paths are obtained. Hence, in general, $H_{k,n}$ for odd k can be decomposed into $\lfloor k/2 \rfloor$ edge-disjoint cycles of length n and $\lfloor n/2 \rfloor$ paths of length *1*.
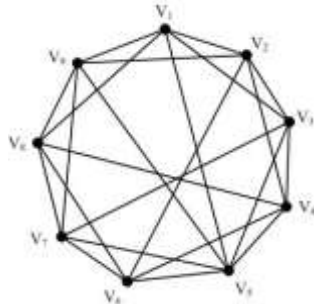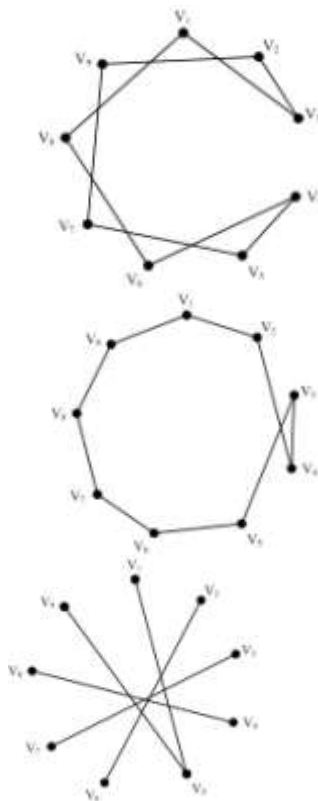

Fig. 6 The Harary Graph $H_{5,9}$

Fig. 7 Hamilton decomposition of $H_{5,9}$

$H_{5,9}$ is decomposed into *2* cycles, each of length *9* and *3* paths of which two are of length *1* and one is of length *2*.
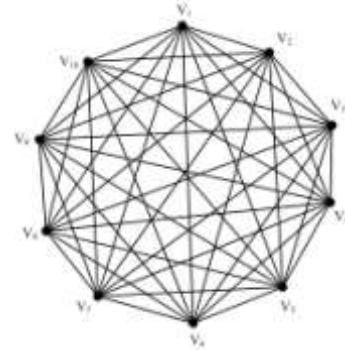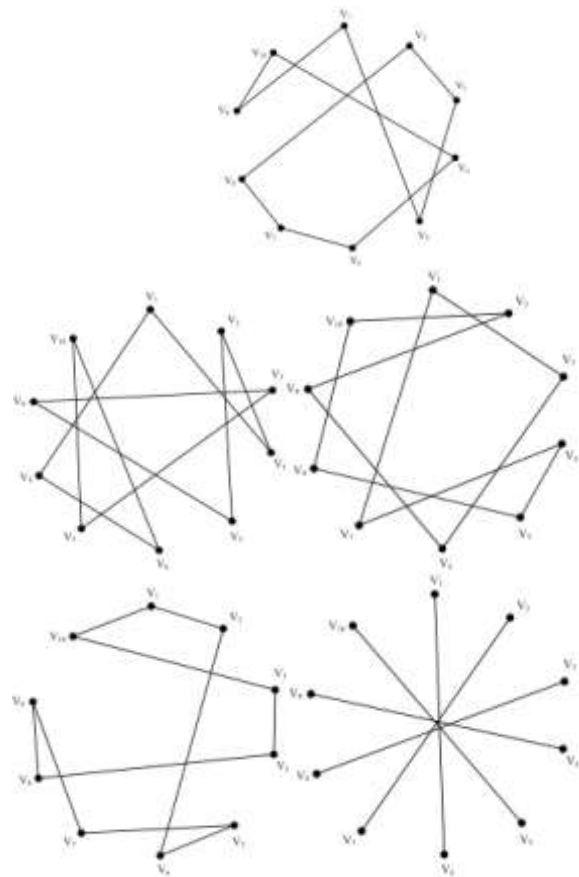

Fig. 8 The Harary Graph $H_{9,10}$

Fig. 9 Hamilton decomposition of $H_{9,10}$

$H_{9,10}$ is decomposed into *4* cycles, each of length *10* and *5* paths, each of length *1*.

IV. CONCLUSION

In this paper we investigated the Hamiltonian decomposition of Harary graphs $H_{k,n}$ for both even and odd k. The Hamilton decomposition of Haray graphs can be put into various applications in the field of networking. Further, alternative proofs to determine the Hamiltonian decomposition of Harary graphs can be studied. We intend to study the

Hamilton decomposition of Generalised Petersen graphs in the future.

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