Lagrange's Interpolation Formula: Representation of Numerical Data by a Polynomial curve

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Abstract: *The interpolation by an idea/method which consists of the representation of numerical data by a suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable leads to the necessity of a formula for representing a given set of numerical data on a pair of variables by a suitable polynomial. One such formula has been developed in this study.* The *formula has been derived from Lagrange's interpolation formula. The formula obtained has been applied to represent the numerical data, on the total population of India since 1971, by a suitable polynomial.*

Keywords: *Interpolation, Lagrange's formula, polynomial curve, representation of numerical data*

INTRODUCTION :

Interpolation is a technique of estimating approximately the value of the dependent variable corresponding to a value of the independent variable lying between its two extreme values on the basis of the given values of the independent and the dependent variables {Hummel (1947), Erdos & Turan (1938) et al}. A number of interpolation formulas namely

(1) Newton's Forward Interpolation formula
(2) Newton's Backward Interpolation formula
(3) Lagrange's Interpolation formula
(4) Newton's Divided Difference Interpolation formula
(5) Newton's Central Difference Interpolation formula
(6) Stirlings formula
(7) Bessel's formula

etc. are available in the literature of numerical analysis {Bathe & Wilson (1976), Jan (1930), Hummel (1947) et al}.

In case of the interpolation by the existing formulae, the value of the dependent variable corresponding to each value of the independent variable is to be computed afresh from the used formula putting the value of the independent variable in it. That is if it is wanted to interpolate the values of the dependent variable corresponding to a number of values of the independent variable by a suitable existing interpolation formula, it is required to apply the formula for each value separately and thus the numerical computation of the value of the dependent variable based on the given data are to be performed in each of the cases. In order to get rid of these repeated numerical computations from the given data, one can think of an approach which consists of the representation of the given numerical data by a suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable. However, a formula is necessary for representing a given set of numerical data on a pair of variables by a suitable polynomial. One such formula has been developed in this study. The formula has been derived from Lagrange's interpolation formula. The formula obtained has been applied to represent the numerical data, on the total population of India since 1971, by a polynomial curve.

2. Lagrange's Interpolation Formula:

Let the dependent variable *Y* assumes the values

$$y_0 , y_1 , y_2 , \ldots\ldots\ldots\ldots\ldots , y_n$$

corresponding to the values

$$x_0 , x_1, x_2, \ldots\ldots\ldots\ldots\ldots x_n$$

of the independent variable *X*.

Lagrange's interpolation formula for interpolating the value of *y* corresponding to a value of *x* between any x_0 and x_n is described by

$$f(x) \text{ or } y_x = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_1)\ldots\ldots(x_0-x_1)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)} y_2 + \ldots\ldots\ldots + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})} y_n \quad (2\text{-}1)$$

{Quadling (1966), Traub (1964), Mills (1977), Revers & Michael (2000), Whittaker & Robinson (1967), Echols (1893) et al}.

3. Representation of Numerical Data by Polynomial Curve:

By algebraic expansion, one can obtain that

$$(x-x_1)(x-x_2) = x^2 - (x_1 + x_2)x + x_1 x_2 = x^2 - \left(\textstyle\sum_{i=1}^{2} x_i\right) x + x_1 x_2 ,$$

Also,

$$(x-x_1)(x-x_2)(x-x_3) = x^3 - (x_1 + x_2 + x_3)\, x^2 + (x_1 x_2 + x_1 x_3 + x_2 x_3)\, x - x_1 x_2 x_3 = x^3 - \left(\textstyle\sum_{i=1}^{3} x_i\right) x^2 + \left(\textstyle\sum_{i=1}^{2}\sum_{j=2}^{3} x_i x_j\right) x - x_1 x_2 x_3 ,$$

Again,

$$(x-x_1)(x-x_2)(x-x_3)(x-x_4) = x^4 - (x_1 + x_2 + x_3 + x_4)\, x^3 + (x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + x_3 x_4)\, x^2 - (x_1 x_2 x_3 + x_1 x_3 x_4 + x_2 x_3 x_4 + x_1 x_2 x_4)\, x + x_1 x_2 x_3 x_4$$

$$= x^4 - \left(\textstyle\sum_{i=1}^{4} x_i\right) x^3 + \left(\textstyle\sum_{i=1}^{3}\sum_{j=2}^{4} x_i x_j\right) x^2 - \left(\textstyle\sum_{i=1}^{2}\sum_{j=2}^{3}\sum_{k=3}^{4} x_i x_j x_k\right) x + x_1 x_2 x_3 x_4 ,$$

Similarly,

$$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5) = x^5 - (x_1 + x_2 + x_3 + x_4 + x_5)\, x^4 + (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_1 x_5 + x_2 x_3 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_4 x_5)\, x^3 - (x_1 x_2 x_3 + x_1 x_3 x_4 + x_1 x_4 x_5 + x_2 x_3 x_4 + x_2 x_3 x_5 + x_3 x_4 x_5 + x_1 x_2 x_5 + x_1 x_3 x_5 + x_2 x_4 x_5)\, x^2 + (x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + x_2 x_3 x_4 x_5 + x_1 x_2 x_4 x_5 + x_1 x_3 x_4 x_5)\, x - x_1 x_2 x_3 x_4 x_5$$

$$= x^5 - \left(\textstyle\sum_{i=1}^{5} x_i\right) x^4 + \left(\textstyle\sum_{i=1}^{4}\sum_{j=2}^{5} x_i x_j\right) x^3 -$$

$(\sum_{i=1}^{3}\sum_{j=2}^{4}\sum_{k=3}^{5} x_i x_j x_k)x^2 +$
$(\sum_{i=1}^{2}\sum_{j=2}^{3}\sum_{k=3}^{4}\sum_{l=4}^{5} x_i x_j x_k x_l) - x_1 x_2 x_3 x_4 x_5$

In general, one can obtain that

$(x-x_1)(x-x_2)\ldots\ldots(x-x_n)$

$= x^n - (\sum_{i=1}^{n} x_i)x^{n-1} + (\sum_{i=1}^{n-1}\sum_{j=2}^{n} x_i x_j)x^{n-2} - (\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} x_i x_j x_k)x^{n-3} + (\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\sum_{k=3}^{n-1}\sum_{l=4}^{n} x_i x_j x_k x_l)x^{n-4} + (-1)^r(\sum_{i_1=1}^{n-r}\sum_{i_2=1}^{n-r+1}\ldots\ldots\sum_{i_r=1}^{r} x_{i_1} x_{i_2}\ldots\ldots x_{i_r})x^{n-r} + (x_1 x_2 x_3 \ldots\ldots x_n)$

$= x^n - (\sum_{i=1}^{n} x_i)x^{n-1} + (\sum_{i=1}^{n-1}\sum_{j=2}^{n} x_i x_j)x^{n-2} - (\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} x_i x_j x_k)x^{n-3} + (\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\sum_{k=3}^{n-1}\sum_{l=4}^{n} x_i x_j x_k x_l)x^{n-4} + (-1)^r(\sum_{i_1=1}^{n-r}\sum_{i_r=1}^{r}\prod_{j=1}^{r} x_{ij})x^{n-r} + + (-1)^n(x_1 x_2 x_3 \ldots\ldots x_n)$

Now, Lagranges interpolation formula, described by equation (2.1) can be expressed as

$$f(x) = \frac{\prod_{i=0,i\neq 0}^{n}(x-x_i)}{\prod_{i=0,i\neq 0}^{n}(x_0-x_i)} f(x_0) + \frac{\prod_{i=0,i\neq 1}^{n}(x-x_i)}{\prod_{i=0,i\neq 1}^{n}(x_1-x_i)} f(x_1) + \ldots\ldots + \frac{\prod_{i=0,i\neq n-1}^{n-1}(x-x_i)}{\prod_{i=0,i\neq n-1}^{n-1}(x_n-x_i)} f(x_n)$$

$$= \sum_{r=0}^{n}\left\{\frac{\prod_{i=0,i\neq r}^{n}(x-x_i)}{\prod_{i=0,i\neq r}^{n}(x_0-x_i)}\right\} f(x_r)$$

$$= \sum_{r=0}^{n}\left\{\frac{f(x_r)}{\prod_{i=0,i\neq r}^{r}(x_0-x_i)}\right\}\prod_{i=0,i\neq r}^{r}(x-x_i) \quad (3.1)$$

The expression (3.1) can be written as

$$f(x) = C_0\{\prod_{i=0, i\neq 0}^{n}(x-x_i)\} + C_1\{\prod_{i=0, i\neq 1}^{n}(x-x_i)\} + C_2\{\prod_{i=0, i\neq 2}^{n}(x-x_i)\} + \ldots\ldots + C_r\{\prod_{i=0}^{n}(x-x_i)\} + C_n\{\prod_{i=0, i\neq n}^{n}(x-x_i)\} \quad (3.2)$$

where $C_0 = \frac{f(x_0)}{\prod_{i=0}^{n}(x_0-x_i)}$, $C_1 = \frac{f(x_1)}{\prod_{i=0}^{n}(x_1-x_i)}$ ……… $i \neq 0, i \neq 1$

……………………

$C_n = \frac{f(x_n)}{\prod_{i=0}^{n-1}(x_{n-1}-x_i)}$ $i \neq n-1$

Now,

$\prod_{i=0, i\neq 0}^{n}(x-x_i)$

$= (x-x_1)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)$

$= x^n + (\sum_{i=0, i\neq 0}^{n} x_i)x^{n-1} + (\sum_{i=0, i\neq 0}^{n}\sum_{j=0, j\neq 0}^{n} x_i x_j)x^{n-2} + (\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} x_i x_j x_k)x^{n-3} + \ldots\ldots + (\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\ldots\ldots\sum_{i_{r-1}=0}^{n} x_{i_1}x_{i_2}\ldots\ldots x_{i_{r-1}})x^{n-r} + \ldots\ldots + (x_1 x_2 x_3 \ldots\ldots x_n)$

$i \neq 0 \quad j \neq 0 \quad k \neq 0$; $i_1 \neq 0, i_2 \neq 0, \ldots\ldots, i_{r-1} \neq 0$

$$\prod_{i=0,i\neq 1}^{n}(x-x_i)$$

$= (x-x_0)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)$

$= x^n + (\sum_{i=0, i\neq 1}^{n} x_i)x^{n-1} + (\sum_{i=0, i\neq 1}^{n}\sum_{j=0, j\neq 1}^{n} x_i x_j)x^{n-2} + (\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} x_i x_j x_k)x^{n-3} + \ldots\ldots +$

$i \neq 1 \quad j \neq 1 \quad k \neq 1$

$(\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \dots\dots\dots \sum_{i_{r-1}=0}^{n} x_{i_1}x_{i_2} \dots\dots x_{i_{r-1}})\, x^{n-r}$

$i_1 \neq 0, i_2 \neq 0, \dots\dots\dots, i_{r-1} \neq 0$

$+ (x_1x_2x_3 \dots\dots\dots x_n)$

$\prod_{i=0, i\neq 2}^{n} (x - x_i)$

$= (x - x_0)(x - x_2)(x - x_3) \dots\dots\dots\dots (x - x_n)$

$= x^n + (\sum_{i=0}^{n} x_i)\, x^{n-1} + (\sum_{i=0}^{n} \sum_{j=0}^{n} x_i x_j)\, x^{n-2}$

$i \neq 2 \qquad i \neq 2 \quad j \neq 2$

$+ (\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} x_i x_j x_k)\, x^{n-3} + \dots\dots\dots +$

$i \neq 2 \quad j \neq 2 \quad k \neq 2$

$(x_1x_2x_3 \dots\dots\dots x_n)$

$\prod_{i=0}^{n} (x - x_i)$

$= (x - x_0)(x - x_1)(x - x_2) \dots\dots\dots\dots (x - x_{n-1})_1)$

$= x^n + (\sum_{i=0}^{n-1} x_i)\, x^{n-1} + (\sum_{i=0}^{n-2} \sum_{j=0}^{n} x_i x_j)\, x^{n-2} +$

$i \neq n \qquad i \neq n \quad j \neq n$

$\sum_{i=0}^{n-3} \sum_{j=0}^{n} \sum_{k=0}^{n} x_i x_j x_k\ )\, x^{n-3} + \dots\dots\dots +$

$i \neq n \quad j \neq n \quad k \neq n$

$(x_1x_2x_3 \dots\dots\dots x_n)$

(3.2) ⇒

$f(x) = C_0\{\, x^n + (\sum_{i=0}^{n} x_i)\, x^{n-1} +$

$(\sum_{i=0}^{n} \sum_{j=0}^{n} x_i x_j)\, x^{n-2} +$

$+(\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} x_i x_j x_k)\, x^{n-3} + \dots +$

$(\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \dots\dots \sum_{i_{r-1}=0}^{n} x_{i_1}x_{i_2} \dots\dots x_{i_{r-1}})\, x^{n-r}$

$i_1 \neq 0,\ i_2 \neq 0, \dots\dots\dots, i_{r-1} \neq 0$

$+ \dots\dots\dots + (x_1x_2x_3 \dots\dots\dots x_n)\,\} +$

$C_1 \{\, x^n + (\sum_{i=0}^{n} x_i)\, x^{n-1} + (\sum_{i=0}^{n} \sum_{j=0}^{n} x_i x_j)\, x^{n-2}$

$i \neq 1 \qquad i \neq 1 \quad j \neq 1$

$+ (\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} x_i x_j x_k)\, x^{n-3} + \dots\dots\dots +$

$i \neq 1 \quad j \neq 1 \quad k \neq 1$

$(\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \dots\dots\dots \sum_{i_{r-1}=0}^{n} x_{i_1}x_{i_2} \dots\dots x_{i_{r-1}})\, x^{n-r}$

$i_1 \neq 0, i_2 \neq 0, \dots\dots\dots, i_{r-1} \neq 0$

$+ (x_1x_2x_3 \dots\dots\dots x_n)\ \} + C_2\{\, x^n + (\sum_{i=0}^{n} x_i)\, x^{n-1}$

$i \neq 2$

$+ (\sum_{i=0}^{n} \sum_{j=0}^{n} x_i x_j)\, x^{n-2} +$

$i \neq 2 \quad j \neq 2$

$(\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} x_i x_j x_k)\, x^{n-3} + \dots\dots\dots +$

$i \neq 2 \quad j \neq 2 \quad k \neq 2$

$(x_1x_2x_3 \dots\dots\dots x_n)\ \}$

$+ \dots\dots\dots\dots +$

$C_n\{\, x^n + (\sum_{i=0}^{n-1} x_i)\, x^{n-1} + (\sum_{i=0}^{n-2} \sum_{j=0}^{n} x_i x_j)\, x^{n-2} +$

$i \neq n \qquad i \neq n \quad j \neq n$

$\sum_{i=0}^{n-3} \sum_{j=0}^{n} \sum_{k=0}^{n} x_i x_j x_k\ )\, x^{n-3} + \dots\dots\dots +$

$i \neq n \quad j \neq n \quad k \neq n$

$(x_1x_2x_3 \dots\dots\dots x_n)\,\}$

$f(x) = [C_0 + C_1 + C_2 + \dots\dots\dots + C_n] + [C_0 (\sum_{i=0}^{n} x_i)$

$i \neq 0$

$+ C_1 (\sum_{i=0}^{n} x_i) + C_2 (\sum_{i=0}^{n} x_i) + \dots\dots\dots +$

$i \neq 1 \qquad i \neq 2$

$C_n (\sum_{i=0}^{n-1} x_i)\ ]\ x^{n-1} +$

$[\, C_0(\sum_{i=0}^{n}\sum_{j=0}^{n} x_i x_j x_j)$

$i \neq 0 \quad j \neq 0$

$i \neq j$

$+ C_1(\sum_{i=0}^{n}\sum_{j=0}^{n} x_i x_j x_j) + C_2(\sum_{i=0}^{n}\sum_{j=0}^{n} x_i x_j x_j)$

$i \neq 1 \quad j \neq 1 \qquad i \neq 2 \quad j \neq 2$

$i \neq j \qquad i \neq j$

$+ \dots\dots\dots + C_n(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j)\ ]\ x^{n-2}$

$+ \dots\dots\dots + [(x_1x_2x_3 \dots\dots x_n) + (x_0\, x_2x_3 \dots\dots x_n)$

$+ \dots\dots\dots + (x_0\, x_2x_3 \dots\dots x_{n-1})\ ]$

Writing $S_r = \sum_{i=0}^{n} x_i$, $S_r\,(p) = \sum_{i=0}^{n}\sum_{j=0}^{n} x_i x_j$

i≠ r i≠ r, j≠ r, i ≠ j

$$S_{r_1 r_2 \ldots\ldots r_p} = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \ldots\ldots\ldots \sum_{i_p=0}^{n} x_{i_1} x_{i_2} \ldots\ldots x_{i_p}$$
$$i_1 \neq r_1, i_2 \neq r_1 \ldots\ldots\ldots i_p \neq r_p$$
$$i_1 \neq i_2 \neq \ldots\ldots\ldots\ldots \neq i_p$$

one can obtain that

$$f(x) = \left(\sum_{i=0}^{n} C_i\right) x^n + \left\{\sum_{i=0}^{n} C_i S_i(1)\right\} x^{n-1} + \left\{\sum_{i=0}^{n} C_i S_i(2)\right\} x^{n-2} + \left\{\sum_{i=0}^{n} C_i S_i(3)\right\} x^{n-3} + \ldots\ldots + \left\{\sum_{i=0}^{n} C_i S_i(r)\right\} x^{n-r} + \left\{\sum_{i=0}^{n} C_i S_i(n-r)\right\} x + \sum_{i=0}^{n} C_i S_i(n) \qquad (3.3)$$

This is the required formula for representation of numerical data by a polynomial curve.

4. An Example of Application of the Formula:

The following table shows the data on total population of India corresponding to the years:

Year	*Total Population*
1971	548159652
1981	683329097
1991	846302688
2001	1027015247
2011	1210193422

Taking 1971 as origin and changing scale by 1/10, one can obtain the following table for independent variable *x* (representing time) and *f*(*x*) (representing total population of India):

Year	x_i	*Total Population*
1971	0	548159652
1981	1	683329097
1991	2	846302688
2001	3	1027015247
2011	4	1210193422

Now here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$

$f(x_0) = 548159652$, $f(x_1) = 683329097$,

$f(x_2) = 846302688$, $f(x_3) = 1027015247$,

$f(x_4) = 1210193422$

Thus,

$$C_0 = \frac{548159652}{(0-1)(0-2)(0-3)(0-4)} = \frac{548159652}{(-1)(-2)(-3)(-4)} = \frac{548159652}{24} = 22839985.5$$

$$C_1 = \frac{683329097}{(1-0)(1-2)(1-3)(1-4)} = \frac{683329097}{1.(-1)(-2)(-3)} = \frac{683329097}{-6} = -113888182.83$$

$$C_2 = \frac{846302688}{(2-0)(2-1)(2-3)(2-4)} = \frac{846302688}{2.1.(-1)(-2)} = \frac{846302688}{4} = 211575672$$

$$C_3 = \frac{1027015247}{(3-0)(3-1)(3-2)(3-4)}$$

$= \frac{1027015247}{3.2.1.(-1)}$

$= \frac{1027015247}{-6}$

$= -171169207.83$

$C_4 = \frac{1210193422}{(4-0)(4-1)(4-2)(4-3)}$

$= \frac{1210193422}{4.3.2.1}$

$= \frac{1210193422}{24}$

$= 50424725.91$

$\therefore \quad C_0 + C_1 + C_2 + C_3 + C_4$

= 22839985.5 + (– 113888182.83) + 211575672 + (–171169207.83) + 50424725.91

= 22839985.5 – 113888182.83 + 211575672 – 171169207.83 + 50424725.91

= 284840383.41 – 285057390.66

= – 217007.25

Again,

$x_1 + x_2 + x_3 + x_4 = 1 + 2 + 3 + 4 = 10$

$x_0 + x_2 + x_3 + x_4 = 0 + 2 + 3 + 4 = 9$

$x_0 + x_1 + x_3 + x_4 = 0 + 1 + 3 + 4 = 8$

$x_0 + x_1 + x_2 + x_4 = 0 + 1 + 2 + 4 = 7$

$x_0 + x_1 + x_2 + x_3 = 0 + 1 + 2 + 3 = 6$

Now,

$x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4$
= 1.2 + 1.3 + 1.4 + 2.3 + 2.4 + 3.4
= 2 + 3 + 4 + 6 + 8 + 12
= 35

$x_0 x_2 + x_0 x_3 + x_0 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4$
= 0.2 + 0.3 + 0.4 + 2.3 + 2.4 + 3.4

= 0 + 0 + 0 + 6 + 8 + 12
= 26

$x_0 x_1 + x_0 x_3 + x_0 x_4 + x_1 x_3 + x_1 x_4 + x_3 x_4$
= 0.1 + 0.3 + 0.4 + 1.3+ 1.4 + 3.4
= 0 + 0 + 0 + 3 + 4 + 12
= 19

$x_0 x_1 + x_0 x_2 + x_0 x_4 + x_1 x_2 + x_1 x_4 + x_2 x_4$
= 0.1 + 0.2 + 0.4 + 1.2 + 1.4 + 2.4
= 0 + 0 + 0 + 2 + 4 + 8
= 14

$x_0 x_1 + x_0 x_2 + x_0 x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3$
= 0.1 + 0.2 + 0.3 + 1.2 + 1.3 + 2.3
= 0 + 0 + 0 + 2 + 3 + 6
= 11

and

$x_1 x_2 x_3$ = 1.2.3 = 6
$x_1 x_3 x_4$ = 1.3.4 = 12
$x_2 x_3 x_4$ = 2.3.4 = 24
$x_1 x_2 x_4$ = 1.2.4 = 8
$x_0 x_2 x_3$ = 0. 2.3 = 0
$x_0 x_2 x_4$ = 0.2.4 = 0
$x_0 x_3 x_4$ = 0.3.4 = 0
$x_0 x_1 x_3$ = 0.1.3= 0
$x_0 x_1 x_4$ = 0.1.4 = 0
$x_0 x_1 x_2$ = 0.1.2 = 0

Also, $x_1 x_2 x_3 x_4$ =1.2.3.4 = 24
$x_0 x_2 x_3 x_4$ = 0.2.3.4 = 0
$x_0 x_1 x_3 x_4$ = 0.1.3.4 = 0
$x_0 x_1 x_2 x_4$ = 0.1.2.4 = 0

Now,

$C_0 (x_1 + x_2 + x_3 + x_4) + C_1 (x_0 + x_2 + x_3 + x_4) +$
$C_2 (x_0 + x_1 + x_3 + x_4) + C_3 (x_0 + x_1 + x_2 + x_4) +$
$C_4 (x_0 + x_1 + x_2 + x_3)$

$= C_0 \times 10 + C_1 \times 9 + C_2 \times 8 + C_3 \times 7 + C_4 \times 6$

= 22839985.5 × 10 + (– 113888182.83) × 9 + 211575672 × 8 + (– 171169207.83) × 7 + 50424725.91 × 6

= 228399855 – 1024993645.5 + 1692605376 – 1198184454.8 + 302548355.46

= 2223553586.5 – 2223178100.3

= 375486.2

$C_0 \ (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) + C_1 \ (x_0 x_2 + x_0 x_3 + x_0 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4\ ) + C_2 \ (x_0 x_1 + x_0 x_3 + x_0 x_4 + x_1 x_3 + x_1 x_4 + x_3 x_4\ ) + C_3 \ (x_0 x_1 + x_0 x_2 + x_0 x_4 + x_1 x_2 + x_1 x_4 + x_2 x_4) + C_4$

$(x_0 x_1 + x_0 x_2 + x_0 x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3)$

= 22839985.5 × 35 + (– 113888182.83) × 26 + 211575672 × 19 + (–171169207.83) × 14 + 50424725.91 ×11

= 799399492.5 – 2961092753.6 + 4019937768 – 2396368909.6 + 554671985.01

= 5374009245.5 – 5357461663.2

= 16547582.3

$C_0 (x_1 x_2 x_3 + x_1 x_3 x_4 + x_2 x_3 x_4 + x_1 x_2 x_4) + C_1 (x_0 x_2 x_3 + x_0 x_2 x_4 + x_0 x_3 x_4 + x_2 x_3 x_4) + C_2 (x_0 x_1 x_3 + x_0 x_3 x_4 + x_0 x_1 x_4 + x_1 x_3 x_4) + C_3 (x_0 x_1 x_2 + x_0 x_2 x_4 + x_0 x_1 x_4 + x_1 x_2 x_4) + C_4 (x_0 x_1 x_2 + x_0 x_1 x_3 + x_0 x_2 x_3 + x_1 x_2 x_3)$

$= C_0 (6 + 12 + 24 + 8) + C_1 (0 + 0 + 0 + 24) + C_2 (0 + 0 + 0 + 12) + C_3 (0 + 0 + 0 + 8) + C_4 (0 + 0 + 0 + 6)$

= 22839985.5 × 50 + (– 113888182.83) × 24 + 211575672 × 12 + (–171169207.83) × 8 + 50424725.91 × 6

= 1141999275 – 2733316387. 9 + 2538908064 –1369353662.6 + 302548355.46

= 3983455694.5 – 4102670050.5

= –119214356

$C_0 (x_1 x_2 x_3 x_4) + C_1 (x_0 x_2 x_3 x_4) + C_2 (x_0 x_1 x_3 x_4) + C_3 (x_0 x_1 x_2 x_4) + C_4 (x_0 x_1 x_2 x_3)$

= 22839985.5 × 24 + (– 113888182.83) × 0 + 211575672 × 0 + (– 171169207.83) × 0 + 50424725.91 × 0

= 548159652

$f(x) = (C_0 + C_1 + C_2 + C_3 + C_4) x^4 - \{ C_0 (x_1 + x_2 + x_3 + x_4) + C_1 (x_0 + x_2 + x_3 + x_4) + C_2 (x_0 + x_1 + x_3 + x_4) + C_3 (x_0 + x_1 + x_2 + x_4) + C_4 (x_0 + x_1 + x_2 + x_3)\} x^3 + \{C_0 (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) + C_1 (x_0 x_2 + x_0 x_3 + x_0 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) + C_2 (x_0 x_1 + x_0 x_3 + x_0 x_4 + x_1 x_3 + x_1 x_4 + x_3 x_4) + C_3 (x_0 x_1 + x_0 x_2 + x_0 x_4 + x_1 x_2 + x_1 x_4 + x_2 x_4) + C_4 (x_0 x_1 + x_0 x_2 + x_0 x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3)\} x^2 - \{C_0 (x_1 x_2 x_3 + x_1 x_3 x_4 + x_2 x_3 x_4 + x_1 x_2 x_4) + C_1 (x_0 x_2 x_3 + x_0 x_2 x_4 + x_0 x_3 x_4 + x_2 x_3 x_4) + C_2 (x_0 x_1 x_3 + x_0 x_3 x_4 + x_0 x_1 x_4 + x_1 x_3 x_4) + C_3 (x_0 x_1 x_2 + x_0 x_2 x_4 + x_0 x_1 x_4 + x_1 x_2 x_4) + C_4 (x_0 x_1 x_2 + x_0 x_1 x_3 + x_0 x_2 x_3 + x_1 x_2 x_3)\} x + C_0 (x_1 x_2 x_3 x_4) + C_1 (x_0 x_2 x_3 x_4) + C_2 (x_0 x_1 x_3 x_4) + C_3 (x_0 x_1 x_2 x_4) + C_4 (x_0 x_1 x_2 x_3)$

$= - 217007.25 x^4 - 375486.2 x^3 + 16547582.3 x^2 - (-119214356) x + 548159652$

$= -217007.25\, x^4 - 375486.2\, x^3 + 16547582.3\, x^2 + 119214356\, x + 548159652$

Thus, the polynomial that can represent the given numerical data is

$$f(x) = -217007.25\, x^4 - 375486.2\, x^3 + 16547582.3\, x^2 + 119214356\, x + 548159652 \quad (3.4)$$

This polynomial yields the values of the function $f(x)$ corresponding to the respective observed values as follows:

$f(0) = -217007.25 \times 0 - 375486.2 \times 0 + 16547582.3 \times 0 + 119214356 \times 0 + 548159652$

$= 548159652$

$f(1) = -217007.25 \times 1 - 375486.2 \times 1 + 16547582.3 \times 1 + 119214356 \times 1 + 548159652$

$= -217007.25 - 375486.2 + 16547582.3 + 119214356 + 548159652$

$= -592493.45 + 683921590.3$

$= 683329097$

$f(2) = -217007.25 \times 2^4 - 375486.2 \times 2^3 + 16547582.3 \times 2^2 + 119214356 \times 2 + 548159652$

$= -217007.25 \times 16 - 375486.2 \times 8 + 16547582.3 \times 4 + 119214356 \times 2 + 548159652$

$= -3472116 - 3003889.6 + 66190329.2 + 238428712 + 548159652$

$= -6476005.6 + 852778693.2$

$= 846302687.6$

$= 846302688$

$f(3) = -217007.25 \times 3^4 - 375486.2 \times 3^3 + 16547582.3 \times 3^2 + 119214356 \times 3 + 548159652$

$= -217007.25 \times 81 - 375486.2 \times 27 + 16547582.3 \times 9 + 119214356 \times 3 + 548159652$

$= -17577587.25 - 10138127.4 + 148928240.7 + 357643068 + 548159652$

$= -27715714.65 + 1054730960.7$

$= 1027015246.1$

$f(4) = -217007.25 \times 4^4 - 375486.2 \times 4^3 + 16547582.3 \times 4^2 + 119214356 \times 4 + 548159652$

$= -217007.25 \times 256 - 375486.2 \times 64 + 16547582.3 \times 16 + 119214356 \times 4 + 548159652$

$= -55553856 - 24031104 + 264761316.8 + 476857424 + 548159652$

$= -79584960 + 1289778392.8 = 1210193432$

5. CONCLUSIONS:

The formula described by equation (3.4) can be used to represent a given set of numerical data on a pair of variables, by a polynomial.

The degree of the polynomial is one less than the number of pairs of observations.

The polynomial that represents the given set of numerical data can be used for interpolation at any position of the independent variable lying within its two extreme values.

The approach of interpolation, described here, can be suitably applied in inverse interpolation also.

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