Notes on Jacobsthal and Jacobsthal-like Sequences

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Abstract — *In this paper, new properties of Jacobsthal and Jacobsthal-like sequences were derived using the method in [1]. Formulas for finding the n^th term of these sequences and solving Jacobsthal mean were presented.*

Keywords — *Jacobsthal sequence, Jacobsthal-like sequence, induction, Jacobsthal mean, missing term*

I. Introduction

Jacobsthal sequence is a recurrence relation with formula

$$J_n = J_{n-1} + 2J_{n-2}$$

with $J_0 = 0$ and $J_1 = 1$ as explained in [2]. This sequence is 0,1,1,3,5,11,21,43,85,171,…(OEIS A001045). This sequence is named after the German Mathematician Ernst Jacobsthal.

Horadam in [3] gave the generating function of this sequence as

$$J_n = \sum_{r=0}^{\frac{n-1}{2}} \binom{n-1-r}{r} 2^r$$

and its Binet form as

$$J_n = \frac{\alpha^n - \beta^n}{3} = \frac{1}{3}[2^n - (-1)^n].$$

In this paper, method of Natividad [1] was used to determine the n^{th} term of Jacobsthal sequence and to solve the Jacobsthal mean.

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II. Main Result

Theorem 2.1 *For any real numbers J_a, J_b, and J_x, the formula for finding the first missing term (mean) of Jacobsthal and Jacobsthal-like sequence is*

$$J_x = \frac{J_b - 2\left[\frac{2^n - (-1)^n}{3}\right] J_a}{\frac{2^{n+1} - (-1)^{n+1}}{3}}$$

where J_x is the first missing term of the sequence, J_a is the first term given, J_b is the last term, and n is the number of missing terms.

Proof. The formula for the first missing term of any Jacobsthal-like sequence can be found and derived using a recognizable pattern. The process of derivation is like in [1] and [4].

From the definition of Jacobsthal sequence, the general formula for J_x will make the calculation easy for the other missing terms. J_x can be solved for Jacobsthal or Jacobsthal-like sequence with

a. One missing term, the sequence is J_a, J_x, J_b

$$2J_a + J_x = J_b$$

then

$$J_x = J_b - 2J_a$$

b. Two missing terms, the sequence is J_a, J_x, J_{x+1}, J_b

$$2J_a + J_x = J_{x+1}$$

$$2J_x + J_{x+1} = J_b$$

then

$$J_x = \frac{J_b - 2J_a}{3}$$

c. Three missing terms, the sequence is $J_a, J_x, J_{x+1}, J_{x+2}, J_b$

$$2J_a + J_x = J_{x+1}$$

$$2J_x + J_{x+1} = J_{x+2}$$

$$2J_{x+1} + J_{x+2} = J_b$$

then

$$J_x = \frac{J_b - 6J_a}{5}$$

d. Four missing terms, the Jacobsthal like sequence is $J_a, J_x, J_{x+1}, J_{x+2}, J_{x+3}, J_b$

$$2J_a + J_x = J_{x+1}$$

$$2J_x + J_{x+1} = J_{x+2}$$

$$2J_{x+1} + J_{x+2} = J_{x+3}$$

$$2J_{x+2} + J_{x+3} = J_b$$

then

$$J_x = \frac{J_b - 10J_a}{11}$$

From the previous equations of J_x, the numerical coefficient of J_a in numerator and the denominator of the formulas were listed in Table 1.

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Table 1. Relationship of number of missing terms with numerator and denominator of formulas.

Number of Missing Term	Coefficient of J_a in numerator	Coefficient of Denominator
1	2	1
2	2	3
3	6	5
4	10	11
.	.	.
.	.	.
.	.	.
n	$2\left[\frac{2^n - (-1)^n}{3}\right]$	$\frac{2^{n+1} - (-1)^{n+1}}{3}$

From Table 1, the formula can be illustrated as

$$J_x = \frac{J_b - 2J_nJ_a}{J_{n+1}}.$$

Using the definition of Jacobsthal sequence as

$$J_n = \left[\frac{2^n - (-1)^n}{3}\right]$$

and by substitution, the theorem is now proved.□

Consequently, we can also find the explicit formula for the n^th term of Jacobsthal or Jacobsthal-like sequence using this method.

Definition 1.1 *The sequence $K_1, K_2, \ldots, K_n$ in which $K_n = 2K_{n-2} + K_{n-1}$ is a Jacobsthal-like sequence.*

Theorem 2.2 *For any real numbers K_1 and K_2, the formula for finding the n^{th} term of Jacobsthal-like sequence is*

$$K_n = 2J_{n-2}K_1 + J_{n-1}K_2$$

where K_n is the n^{th} term of Jacobsthal-like sequence, K_1 is the first term, K_2 is the second term and J_{n-1}, J_{n-2} are the corresponding Jacobsthal numbers.

Proof. The numerical coefficients for the first two terms of the sequence were listed. Equations were solved for $3 \le n \le 6$ and the coefficients were tabulated like in [5] and [6].

Table 2. Coefficients of K_1 and K_2 in each equation.

N	K_1	K_2
3	2	1
4	2	3
5	6	5
6	10	11

It is interesting to note that the coefficient of K_1 corresponds to twice of the n-2^th term of Jacobsthal sequence while K_2 corresponds to the n-1^th term of the Jacobsthal sequence. So, we can conclude that the n^{th} term (K_n) is equal to $2J_{n-2}K_1 + J_{n-1}K_2$ completing the proof. □

The formula can be validated in any values of n by using mathematical induction. The formula can be easily verified using n = 3, 4, 5 and so on. Let P(n) as

$$K_n = 2J_{n-2}K_1 + J_{n-1}K_2$$

then P(m) is

$$K_m = 2J_{m-2}K_1 + J_{m-1}K_2.$$

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It also follows that P(m+1) is

$$K_{m+1} = 2J_{m-1}K_1 + J_mK_2.$$

The assumption of P(m) must imply the truth of P(m+1) to verify the formula. In this process, we will add $2K_{m-1}$ to both sides of P(m). The equation will become

$$2K_{m-1} + K_m = 2J_{m-2}K_1 + J_{m-1}K_2 + 2K_{m-1}$$

But since $K_{m-1} = 2J_{m-3}K_1 + J_{m-2}K_2$ and $2K_{m-1} + K_m = K_{m+1}$,

$$K_{m+1} = 2J_{m-2}K_1 + J_{m-1}K_2 + 2(2J_{m-3}K_1 + J_{m-2}K_2)$$

$$K_{m+1} = 2J_{m-2}K_1 + J_{m-1}K_2 + 4J_{m-3}K_1 + 2J_{m-2}K_2$$

$$K_{m+1} = 2(J_{m-2}K_1 + 2J_{m-3}K_1) + J_{m-1}K_2 + 2J_{m-2}K_2$$

Further, $J_{m-2}K_1 + 2J_{m-3}K_1 = J_{m-1}K_1$ and $J_{m-1}K_2 + 2J_{m-2}K_2 = J_mK_2$ which makes our equation becomes

$$K_{m+1} = 2J_{m-1}K_1 + J_mK_2$$

The resulting equation is exactly our P(m+1), hence, the formula is valid for any value of n.

III. CONCLUSIONS

Explicit formulas for solving n[th] term of Jacobsthal sequence and finding Jacobsthal mean were presented in this paper. The method used in this paper may be extended to other recursive sequences

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