On Pseudo Ricci-Symmetric $N(k)$ −Contact Metric Manifold

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Abstract: *The present paper deals with the study of pseudo Ricci-symmetric $N(k)$ −contact metric manifold. Here we consider generalized pseudo-Ricci symmetric, almost pseudo Ricci-symmetric and ϕ-pseudo Ricci-symmetric $N(k)$ −contact metric manifold and obtained some interesting results.*

Keywords: *N(k)-contact metric manifold, generalized pseudo-Ricci symmetric, almost pseudo Ricci-symmetric, ϕ-pseudo Ricci-symmetric, η-Einstein manifold*.

1. Introduction

The study of Riemann symmetric manifolds began with the work of Cartan [8]. Accordingto Cartan, a Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies the relation $DR = 0$, where D is the covariant differentiation operator with respect to the metric tensor g. During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent. manifold [25], semi symmetric manifold [21], pseudo symmetric manifold [9, 12] etc.,

If the Ricci tensor S of type $(0,2)$ in a Riemannian manifold M satisfies the relation $DS = 0$, then S is said to be Ricci symmetric. The notion of Ricci symmetry has been studied extensively by many authors in several ways to a different extent viz., Ricci recurrent manifold [18], Ricci semi symmetric manifold [21], Ricci pseudo symmetric manifold [10, 13], weakly Ricci symmetric manifold [23].

A (2n+1)-dimensional non-flat Riemannian manifold M is said to be pseudo Ricci symmetric if its Ricci tensor S of type (0,2) is not identically zero and satisfies the relation

$$(D_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y),$$

for any vector fields X, Y and Z, where A is a nowhere vanishing 1-form on M. The pseudo Ricci symmetric manifolds have also been studied by Arslan et. al [1], De and Mazumder [14], De et. al [15] and many others. According to Tamassy and Binh [23], weakly symmetric and weakly Ricci-symmetric manifolds are generalization of pseudo symmetric and pseudo Ricci-symmetric manifolds respectively. This type of manifolds were studied extensively by several authors with different structures viz., [17,19,16,20] etc.,

We organized the paper as follows: In Section 2, we recall some definitions and basic facts concerning $N(k)$ −contact metric manifolds which were used throughout the paper. In Section 3, we study generalized pseudo-Ricci symmetric $N(k)$-contact metric manifold and it is shown that the relation $2A + B + C$ is always zero. And in Section 4, we describe almost pseudo Ricci symmetric $N(k)$-contact metric manifold and obtain that the sum $3A + B$ is everywhere zero and there exists no proper pseudo Ricci-symmetric $N(k)$-contact metric manifold. Finally, in Section 5, we consider ϕ-pseudo Ricci-symmetric $N(k)$-contact metric manifold. We prove that a ϕ-Ricci symmetric $N(k)$-contact metric manifold is η-Einstein and that the Sasakian manifold reduces to Einstein.

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold M is said to be Contact manifold if it carries a global differentiable 1-form η which satisfies the condition $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Also a Contact manifold admits an almost Contact structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a characteristic vector field and η is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi,$$
$$\eta(\xi) = 1, \ \phi\xi = 0, \ \eta(\phi) = 0. \tag{2.1}$$

An almost Contact structure is said to be normal if the induced almost complex structure J on the product manifold $M \otimes R$ defined by

$$J\left(X,\lambda\frac{d}{dt}\right) = (\phi X - \lambda\xi, \eta(X)\frac{d}{dt}),$$

is integrable, where X is tangent to M, t is the coordinate of R and λ is a smooth function on $M \times R$. The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor $[\phi,\phi] + 2d\eta\otimes\xi$, where $[\phi,\phi]$the Nijenhuis tensor of is ϕ. Let g be the compatible Riemannian metric with almost Contact structure (ϕ,ξ,η) that is,

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$g(X,\xi) = \eta(X) \qquad (2.2)$$

for all vector fields $X, Y \in \chi(M)$. A manifold M together with this almost Contact metric structure is said to be almost Contact metric manifold and it is denoted by $M(\phi,\xi,\eta,g)$. An almost Contact metric structure reduces to a contact metric structure if $g(X,\phi Y) = d\eta(X,Y)$. Moreover, if D denotes the Riemannian connection of g, then the following relation holds:

$$D_X\xi = -\phi X - \phi hX. \qquad (2.3)$$

Blair, Koufogiorgos and Papantoniou [5] introduced the (k,μ) −nullity distribution of a Contact metric manifold M and is defined by

$$N(k,\mu): p \to Np(k,\mu)\ \{U \in TpM \mid R(X,Y)U = (kI + \mu h)g(Y,U)X - g(X,U)Y\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [4]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [3].

In a N(k)-contact metric manifold the following relations hold:

$$h^2 = (k-1)\phi^2 \qquad (2.4)$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \qquad (2.5)$$
$$S(X,Y) = [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \qquad (2.6)$$
$$S(\phi X,\phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y), \qquad (2.7)$$
$$S(X,\xi) = 2nk\eta(X), \qquad (2.8)$$
$$(D_X\eta)(Y) = g(X + hX, \phi Y). \qquad (2.9)$$

Definition 2.1. A $(2n+1)$-dimensional N(k)-contact metric manifold M is said to be η- Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

For any vector fields X and Y, where a and b are constants. If $b = 0$ then the manifold M is an Einstein manifold.

3. Generalized pseudo-Ricci symmetric $N(k)$-contact metric manifold

Let M be a $(2n+1)$-dimensional generalized pseudo-Ricci symmetric $N(k)$-contact metric manifold. Then by definition, we have

$$(D_XS)(Y,Z) = 2A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(X,Y) \qquad (3.1)$$

where A, B and C are three non-zero 1-forms.

Putting $Z = \xi$ in (3.1) and using (2.8), we get

$$2nkg(X,\phi Y) + 2nkg(hX,\phi Y) + S(Y,\phi X) + S(Y,\phi hX) = 4nkA(X)\eta(Y) + 2nkB(Y)\eta(X) + C(\xi)S(X,Y) \qquad (3.2)$$

Again putting $X = Y = \xi$ in (3.2), yields

$$2A(\xi) + B(\xi) + C(\xi) = 0 \qquad (3.3)$$

Taking $X = \xi$ in (3.2), we have

$$\eta(Y)[2A(\xi) + C(\xi)] + B(Y)] = 0 \qquad (3.4)$$

By virtue of (3.3), equation (3.4) becomes

$$B(Y) = \eta(Y)B(\xi). \qquad (3.5)$$

Similarly by taking $Y = \xi$ in (3.2) and using (3.3), we get

$$A(X) = \eta(X)A(\xi). \qquad (3.6)$$

Substituting $X = Y = \xi$ in (3.1) and by virtue of (3.3), we obtain

$$C(Z) = C(\xi)\eta(Z). \qquad (3.7)$$

In view of (3.5), (3.6) and (3.7), we have

$$2A(X) + B(X) + C(X) = 0 \text{ for all X..}$$

Hence we can state the following theorem:

Theorem 3.1. A $(2n+1)$-dimensional $N(k)$-contact metric manifold is generalized pseudo Ricci-symmetric, unless the sum of $2A, B$ and C is everywhere zero.

4. Almost pseudo Ricci-symmetric N (k)-contact metric manifold

In 2007, Chaki and Kawaguchi [11] introduced a type of non-flat Riemannian manifold whose Ricci tensor S of type (0, 2) satisfies the condition

$$(D_XS)(Y,Z) = [A(X) + B(X)]S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y). \qquad (4.1)$$

Where A, B non-zero 1-forms are called the associated 1-forms and D denotes the operator of covariant differentiation with respect to the metric g. Such a manifold is called an almost pseudo Ricci symmetric manifold. If in particular $A = B$ then the manifold becomes a pseudo Ricci symmetric manifold introduced by Chaki [10].

Let us consider an almost pseudo Ricci-symmetric $N(k)$-contact metric manifold M. Now putting $Z = \xi$ in (4.1), we get

$$(D_XS)(Y,\xi) = [A(X) + B(X)]S(Y,\xi)$$

$$+A(Y)S(X,\xi)+A(\xi)S(X,Y) \quad (4.2)$$

By using (2.3) and (2.9) in (2.8), we have

$$(D_X S)(Y,\xi) = -2nkg(\phi X+\phi hX, Y) + S(\phi X+\phi hX, Y). \quad (4.3)$$

By virtue of (2.8) and (4.3), (4.2) becomes

$$-2nkg(\phi X+\phi hX,Y)+S(\phi X+\phi h) = 2nk[A(X)+B(X)]\eta(Y) + 2nkA(Y)+A(\xi)\eta(Y). \quad (4.4)$$

Putting $X=\xi$ in (4.4) and using (2.1), we get

$$2nk\big[\big(2A(\xi)+B(\xi)\big)\eta(Y)+A(Y)\big]=0 \quad (4.5)$$

Next putting $Y=\xi$ in (4.5) and by virtue of (2.5), we get

$$2nk[3A(\xi)+B(\xi)]=0.$$

Since $2nk\neq 0$, we get from above that

$$3A(\xi)+B(\xi)=0. \quad (4.6)$$

Again putting $Y=\xi$ in (4.4) and then using (2.1) and (2.8) yields

$$2nk[A(X)+B(X)+2A(\xi)\eta(X)]=0. \quad (4.7)$$

Now replacing Y by X in (4.5) and adding with (4.7), and by virtue of (4.6), we get

$$2nk[A(\xi)\eta(X)+2A(X)+B(X)]=0. \quad (4.8)$$

Again replacing Y by X in (4.5) and adding with (4.8), and in view of (4.6), we get

$$2nk[3A(X)+B(X)]=0 \quad (4.9)$$

Hence we can state the following:

Theorem 4.2. An $(2n+1)$-dimensional $N(k)$-contact metric manifold is almost pseudo Ricci symmetric, if the sum $3A+B$ is everywhere zero.

If in particular $B=A$, then the manifold reduces to a pseudo Ricci-symmetric and from (4.9) we get $A=0$ which is not possible from the definition of pseudo Ricci-symmetric manifold.

Thus we have the following corollary:

Corollary 4.1. There exists no proper pseudo Ricci-symmetric $N(k)$-contact metric manifold.

5. ϕ-pseudo Ricci-symmetric $N(k)$-contact metric manifold

Definition 5.2. An $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be ϕ-pseudo Ricci-symmetric if the Ricci operator Q satisfies

$$\phi^2\big((D_XQ)(Y)\big)=2A(X)Q(Y)+A(Y)Q(X) + S(X,Y)\rho \quad (5.1)$$

for any vector fields X and Y and A is a non zero 1-form defined by

$$A(X)=g(X,\rho), \quad (5.2)$$

Where ρ is the vector field associated to the 1-form A.

If, in particular, $A=0$, then (5.1) turns into the notion of ϕ-Ricci symmetric $N(k)$-contact metric manifold introduced by Shukla and Shukla [22].

Let us consider a ϕ pseudo Ricci-symmetric $N(k)$-contact metric manifold. Then by virtue of (2.1) we have

$$-D_XQ(Y)+QD_XY+\eta\big((D_XQ)(Y)\big)\xi = 2A(X)Q(Y)+A(Y)Q(X)+S(X,Y)\rho. \quad (5.3)$$

Taking inner product of (5.3) with respect to Z, we get

$$-g(D_XQ(Y),Z)+S(D_XY,Z) + \eta\big((D_XQ)(Y)\big)\eta(Z)=2A(X)S(Y,Z) + A(Y)S(X,Z)+S(X,Y)A(Z) \quad (5.4)$$

Putting $Y=\xi$ in (5.4) and using (2.3) and (2.8), we have

$$2nkg(\phi X,Z)+2nkg(\phi hX,Z)-S(\phi X,Z) - S(\phi hX,Z)=4nkA(X)\eta(Z)+A(\xi)S(X,Z) + 2nkA(Z)\eta(X) \quad (5.5)$$

Replacing X by ϕX in (5.5), gives

$$-2nkg(X,Z)+2nkg(hX,Z)+S(X,Z) - S(hX,Z)4nkA(X)\eta(Z)+A(\xi)S(\phi X,Z) \quad (5.6)$$

In view of (2.6) and (2.8), equation (5.6) turns into

$$\frac{2nk+(n+1)\mu}{2(n-1)+\mu}S(X,Z)=Mg(X,Z) + [(k-1)(2(n-1)+\mu)]\eta(X)\eta(Z) + 4nkA(\phi X)\eta(Z)+A(\xi)S(\phi X,Z), \quad (5.7)$$

where $M=2nk+\frac{(2(n-1)-n\mu)[2nk-2(n-1)-n\mu]}{2(n-1+\mu)} -(k-1)(2(n-1)+\mu)$

This leads to the following result:

Theorem 5.3. In a ϕ-pseudo Ricci-symmetric $N(k)$-contact metric manifold, the Ricci tensor S is of the form (5.7).In particular if $A=0$, then from (5.7) we have

$$S(X,Z)=\frac{(2(n-1)+\mu)M}{2nk+(n+1)\mu}g(X,Z) + \frac{(2(n-1)+\mu)(k-1)(2(n-1)+\mu)}{2nk+(n+1)\mu}\eta(X)\eta(Z). \quad (5.8)$$

Hence we can state the following Corollaries:

Corollary 5.2. [22] A ϕ-Ricci symmetric $N(k)$-contact metric manifold is an η-Einstein manifold.

Corollary 5.3. [4] If $k=1$ then from (5.8), a ϕ-Ricci symmetric Sasakian manifold is reduces to Einstein manifold.

6. References

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