Assosymmetric Rings with Weak Novikov Identity

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ABSTRACT

In this paper we show that in a non-associative 2-and 3-divisible prime assosymmetric ring R satisfying the weak Novikov identity (w,x,yz) = y (w,x,z), the square of every element of R is in the nucleus and then the non-zero idempotent e in R is the identity element of R.

INTRODUCTION

Right alternative rings satisfying the weak Novikov identity are studied in [3] and it is shown that the square of every element of the ring is in the nucleus. Paul [4] proved that if R is a prime non-associative ring satisfying (x,y,z) = (x,z,y) and with commutators in the left nucleus, then a non-zero idempotent e is the identity element of R if and only if ***e*** belongs to the nucleus. Now we show that in a non-associative 2-and 3-divisible prime assosymmetric ring R satisfying the weak Novikov identity, the square of every element of R is in the nucleus and the non-zero idempotent ***e*** in R is the identity element of R.

PRELIMINARIES

The commutator (x,y) of two elements x and y in a ring is defined by (x,y) = xy - yx. The associator (x,y,z) is defined by (x,y,z) = (xy)z - x(yz), for all x,y,z in a ring. An assosymmetric ring R is a non-associative ring in which (x,y,z)=(P(x), P(y), P(z)) for each permutation P of x, y and z. These rings are neither flexible nor power-associative. The nucleus N in R is the set of elements n $\in$ R such that (n,x,y) = (x,n,y) = (x,y,n) = 0 for all x,y in R. The center C of R is the set of elements c $\in$ N such that (c,x) = 0 for all x, y in R. Let I be the associator ideal of R. I consists of the smallest ideal which contains all associators. A non-associative ring is called prime if for any two of its ideals A and B with AB = (0), it follows that either A = (0) or B = (0). R is called k - divisible if kx = 0 implies x=0,x $\in$ R and k is a natural number. A ring R is said to have a Peirce decomposition relative to the idempotent ***e*** $\in$ R if R can be decomposed into a direct sum of R_{ij} (i,j=0,1) where R_{ij} ={x$\in$R / xe = jx and ex = ix).

In an arbitrary ring the following identities hold:

(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z ....(1)

f(w,x,y,z) = (wx,y,z) – x(w,y,z) - (x,y,z)w

(x,y,z) + (y,z,x) + (z,x,y) = (xy,z) + (yz,x) + (zx,y) ....(2)

And

(xy,z) - x(y,z) - (x,z)y = (x,y,z) - (x,z,y) + (z,x,y) ....(3)

Putting z = x in (3) gives

(xy,x) + x(x,y) = (x,y,z) ....(4)

In any assosymmetric ring (3) becomes :

$(xy,z) - x(y,z) - (x,z)y = (x,y,z)$ ....(5)

It is proved in (1) that in a 2- and 3- divisible assosymmetric ring R the following identities hold for all w,x,y,z,t in R :

$f(w,x,y,z) = 0$,That is: $(wx,y,z) = x(w,y,z) + (x,y,z)w$ ....(6)

$((w,x),y,z) = 0$ ....(7)

And

$((w,x,y),z,t) = 0$ ....(8)

That is, every commutator and associator is in the nucleus N. Suppose that $n\in N$, Then with w = n in (1) we get $(nx,y,z) = n(x,y,z)$.Combining this with (7) yields :

$(nx,y,z) = n(x,y,z) = (xn,y,z)$ ....(9)

From (7), we have :

$(R, N) \subseteq N$ ....(10)

MAIN RESULTS

LEMMA: If R is a 2- and 3- divisible prime assosymmetric ring, then all commutators are in the center.

Proof : By forming associators on each side of (4) and (7) :

$((x,y,x),r,s) = (x(x,y),r,s) = ((x,y)x,r,s)$

Using (1) and (7) we have :

$((x,y)x,r,s) = (x,y)\,(x,r,s)$

We conclude that :
$((x,y,x),r,s) = (x,y)\,(x,r,s)$

Linearizing (replacing x by x+t), we obtain :

$((x,y,t) + (t,y,x),r,s) = (x,y)(t,r,s) + (t,y)(x,r,s)$.

If we substitute a commutator v for t, we see that $(v,y)\,(x,r,s) = 0$.

This can be restated as :

$((R,R),R)\,(R,R,R)) = 0$. But now the ideal generated by double commutators $((R,R),R)$ (which can be characterized as all sums of double commutators plus right multiples of double commutators, because of (7)) annihilates the associator ideal. Since R is prime and not associative

We conclude $((R,R),R) = 0$ ....(11)

Thus the commutators are in the center.

By forming the commutators of each side of (2) with w and using (11) it follows that

$3((x,y,z),w) = 0$

Thus $((x,y,z),w) = 0$

THEOREM 1 : If R is a non-associative 2-and 3-divisible assosymmetric ring satisfying weak Novikov identity :
$(w,x,yz) = y(w,x,z)$ ...(12)

then x^2 is in the nucleus N.

Proof : By taking $w = x$ in (6), we get

$$(x^2,y,z) = x(x,y,z) + (x,y,z)x \quad ....(13)$$

In an assosymmetric ring we have :

$(x^2,y,z) = (y,z,x^2)$

On the other hand (12) implies that :

$(y,z,x^2) = x(y,z,x) = x(x,y,z)$

Thus from (13) we must have $(x,y,z)x = 0$

Since $((x,y,z),x) = 0$ and $(x,y,z)x = 0$,

We have $x(x,y,z) = 0$, so that using (13),

We get $(x^2,y,z) = 0$.

Therefore x^2 is in the nucleus N.

We use the proof of theorem 2[4], to prove the next theorem.

THEOREM 2 **:** If R is a non-associative 2- and 3- divisible prime assosymmetric ring satisfying the weak Novikov identity, then the non-zero idempotent ***e*** in R is the identity element of R.

Proof : From theorem 1, $\boldsymbol{e} \in N$. By lemma 1 in Rich[5], we have a decomposition $R = \oplus R_{ij}$, i,j = 0,1 relative to ***e*** with $R_{ij} R_{kl} \subseteq \delta_{jk} R_{il}$ (δ denotes the kronecker delta). Now $R_{10} = (e, R_{10}) = - (R_{10} , e)$ and $R_{01} = (R_{01},e)$.Since $\boldsymbol{e} \in N$ and $(R,N) \subseteq N$, R_{10} and $R_{01} \subseteq N$. Now N is a subring of R. It follows that $R_{10} R_{01}+ R_{01} R_{10} \subseteq N$. This, together with the property $R_{ij} R_{kl} \subseteq \delta_{jk} R_{il,}$ allows us to conclude that :
$B = R_{10} R_{01}+ R_{10}+ R_{01}+ R_{01} R_{10}$ is an ideal of R contained in N. Let I be the associator ideal of R.We shall show that $BI = (0)$. Let $b \in B$,

Then using (1), we get :
$(bx,y,z) - (b,xy,z) + (b,x,yz) = b(x,y,z) + (b,x,y)z$
Since B is an ideal contained in N and $b\in B$ we have :
$(bx,y,z) = (b,xy,z) = (b,x,yz) = (b,x,y) = 0$
Thus, from the above equation, we get $b(x,y,z) = 0$.
Also, since $b \in N$, we get :
$b((x,y,z)w) = (b(x,y,z))w = 0$

Thus we have proved that $bI = (0)$ for all b in B. Hence $BI = (0)$. But R is prime and non-associative . This implies that $B = (0)$. So we have $R = R_{11} \oplus R_{00.}$

Thus, R_{11} and R_{00} are ideals of R such that $R_{11}R_{00} = (0)$.

From the primeness of R again $R_{11} = (0)$ or $R_{00} = (0)$. But $0 \neq \boldsymbol{e} \in R_{11}$ so that $R_{11} \neq (0)$. We must have $R_{00} = (0)$. This implies that $\boldsymbol{e}$ is the identity element of R.

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