A Study of W4 Semi-Symmetric Generalized Sasakian-Space-Form

Abhishek Singh

Assistant Professor, Department of Mathematics, Babu Banarasi Das University, Lucknow, U.P, India,

Abstract: *In this paper, we study W_4 semi-symmetric generalized Sasakian-space-form. The Ricci-tensor, the Ricci-operator and the scalar curvature are also found in a W_4 semi-symmetric generalized Sasakian-space-form.*

Keywords: *Sasakian-space-form, W_4 curvature tensor, Ricci-tensor, Ricci-operator and scalar curvature.*

I. Introduction

Generalized Sasakian-space-form was introduced by Alegre et al. [1] and studied the notion of generalized Sasakian-space-form. A generalized Sasakian-space-form is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = {} & f_1[g(Y,Z)\,X - g(X,Z)\,Y] \\ & + f_2[g(X,\varphi Z)\,\varphi Y - g(Y,\varphi Z)\,\varphi X + 2g(X,\varphi Y)\,\varphi Z] \\ & + f_3[\eta(X)\,\eta(Z)\,Y - \eta(Y)\,\eta(Z)\,X \\ & + g(X,Z)\,\eta(Y)\,\xi - g(Y,Z)\,\eta(X)\,\xi], \end{aligned} \tag{1.1}$$

where f_1, f_2 and f_3 are differentiable functions on M and X, Y, Z are vector fields on M. We shall write generalized Sasakian-space-form as $M\,(f_1, f_2, f_3)$ in such case. This kind of manifold looks as a natural generalization of Sasakian-space-form $M\,(c)$, which can be obtain as a particular case of generalized Sasakian-space-form by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where c denotes the constant $\varphi-$ sectional curvature. Moreover, Kenmotsu space forms, cosympletic space form, are also particular cases of generalized Sasakian-space-form $M\,(f_1, f_2, f_3)$. Contact metric and trans-Sasakian generalized Sasakian-space-forms are also studied by Alegre and Carriazo [2]. Conformally flat and locally symmetric generalized Sasakian-space-form is studied by Kim [4] in his paper.

In this present paper W_4 semi-symmetric generalized Sasakian-space form has been studied studied. G. P. Pokhariyal [5] introduced the notion of W_4 curvature tensor. A $(2n+1)-$dimensional Riemannian M is W_4 flat if $W_4 = 0$, where W_4 curvature tensor is defined as

$$W_4(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[g(X,Z)QY - g(X,Y)QZ], \tag{1.2}$$

where Q is the field of symmetric endomorphism corresponding to the Ricci tensor S i.e. $g(QX,Y) = S(X,Y)$.

If a Riemannian manifold satisfies $R(X,Y)W_4 = 0,$ where W_4 is a W_4 curvature tensor, then the manifold is said to be W_4 semi-symmetric manifold.

II. Preliminaries

We recall some definitions and basic formulas in this section which will use later. For this, we recommend the reference [3]. A $(2n+1)-$dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field φ, a unique global non-vanishing structural vector field ξ (called the vector field) and a $1-$form η such that

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$d\eta(X,\xi) = 0, \qquad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$d\eta(X,Y) = g(X,\varphi Y), \qquad \eta o \varphi = 0. \tag{2.4}$$

Such a manifold is called contact manifold if $\eta \wedge (d\eta)^n \neq 0,$ where n is n^{th} exterior power. For contact manifold we also have $d\eta = \Phi$, where $\Phi(X,Y) = g(\varphi X, Y)$ is called fundamental $2-$form on M. If ξ is killing vector field, then M is said to be $K-$ contact manifold. The almost contact metric structure (φ, ξ, η, g) on M is said to be normal if

$$[\varphi, \varphi](X,Y) + 2d\eta\,(X,Y)\xi = 0, \tag{2.5}$$

for all vector field X, Y on M, where $[\varphi, \varphi]$ denotes the Nijenhuis tensor of φ given by

$$[\varphi, \varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.6}$$

An almost contact metric manifold M is said to be $\eta-$Einstein if its Ricci-tensor S is of the form

$$S(X,Y) = cg(X,Y) + d\eta(X)\,\eta(Y), \tag{2.7}$$

where c and d are smooth functions on M. A $\eta-$Einstein manifold becomes Einstein if $d = 0.$

If $\{e_1, e_2, ..., e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in an almost contact metric manifold M of dimension $(2n+1)$, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi e_i) = 2n, \tag{2.8}$$

$$\begin{aligned}\sum_{i=1}^{2n} g(e_i, Y)\, S(X, e_i) &= \sum_{i=1}^{2n} g(\varphi e_i, Y)\, S(X, \varphi e_i) \\ &= S(X,Y) - S(X,\xi)\,\eta(Y),\end{aligned} \tag{2.9}$$

for all $X, Y \in T(M)$. In view of (2.4) and (2.9) and we have

$$\sum_{i=1}^{2n} g(e_i, \varphi Y)\, S(\varphi X, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi Y)\, S(\varphi X, \varphi e_i)$$
$$= S(\varphi X, \varphi Y). \quad (2.10)$$

III. Some results on generalized Sasakian-space –form

For a generalized Sasakian-space–form $M\,(f_1, f_2, f_3)$ of dimension $(2n+1)$, we have

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)\,X - \eta(X)\,Y], \quad (3.1)$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)\,g(X,Y)$$
$$-(3f_2 + (2n-1)\,f_3)\,\eta(X)\,\eta(Y). \quad (3.2)$$

F (3.1), we get

$$R(X,\xi)\,\xi = (f_1 - f_3)[X - \eta(X)\,\xi], \quad (3.3)$$

$$R(X,\xi)\,Y = (f_1 - f_3)(\eta(Y)\,X - g(X,Y)\,\xi), \quad (3.4)$$

$$Q(X) = (2nf_1 + 3f_2 - f_3)\,X - (3f_2 + (2n-1)\,f_3)\,\eta(X)\,\xi. \quad (3.5)$$

$$r = 2n(2n+1)\,f_1 + 6nf_2 - 4nf_3, \quad (3.6)$$

where Q denotes Ricci operator and r is said to be scalar curvature of $M\,(f_1, f_2, f_3)$. From (3.2) and (3.5), we get

$$S(X,\xi) = 2n(f_1 - f_3)\,\eta(X) \quad (3.7)$$

and

$$Q\xi = 2n(f_1 - f_3)\,\xi. \quad (3.8)$$

from (3.7), we have

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi e_i, \varphi e_i)$$
$$= r - 2n(f_1 - f_3), \quad (3.9)$$

where r is scalar curvature. In a generalized Sasakian-space-form $M\,(f_1, f_2, f_3)$, we have

$$R(X,\xi,\xi,Y) = R(\xi, X, Y, \xi)$$
$$= (f_1 - f_3)\,g(\varphi X, \varphi Y), \quad (3.10)$$

and

$$\sum_{i=1}^{2n} R(e_i, X, Y, e_i) = \sum_{i=1}^{2n} R(\varphi e_i, X, Y, \varphi e_i)$$
$$= S(X,Y) - (f_1 - f_3)\,g(\varphi X, \varphi Y), \quad (3.11)$$

for all $X, Y \in T(M)$.

IV. W_4 semi-symmetric generalized Sasakian- space-form

Let $M\,(f_1, f_2, f_3)$ be a $(2n+1)-$dimensional generalized Sasakian-space-form. We obtain from equation (1.2) by using equations (2.2), (3.1) and (3.7)

$$\eta\,(W_4(X,Y)\,Z)(f_1-f_3)\,[\eta(X)\,g(Y,Z)-\eta(Z)\,g(X,Y)]. \tag{4.1}$$

On taking $Y=\xi$ in the equation (4.1), we get

$$\eta\,(W_4(X,\xi)\,Z)=0. \tag{4.2}$$

The condition of quasi-conformally semi-symmetric manifold is

$$R(X,Y).W_4=0. \tag{4.3}$$

In virtue of above equation, we get

$$\begin{aligned}&R(X,Y)\,W_4(U,V)\,W-W_4(R(X,Y)U,V)\,W\\&-W_4(U,R(X,Y)\,V)W-W_4(U,V)\,R(X,Y)\,W=0,\end{aligned} \tag{4.4}$$

Operating η both side and putting $X=\xi$ in (4.4), we have

$$\begin{aligned}&g(R(\xi,Y)W_4(U,V)\,W,\xi)-g(W_4(R(\xi,Y)\,U,V)\,W,\xi)\\&-g(W_4(U,R(\xi,Y)\,V)\,W,\xi)-g(W_4(U,V)\,R(\xi,Y)\,W,\xi)=0.\end{aligned} \tag{4.5}$$

Using (2.3), (3.4) and (3.10) in above equation, we get

$$\begin{aligned}&(f_1-f_3)[W_4(U,V,W,Y)-\eta(Y)\,\eta(W_4(U,V)W)\\&+\eta(U)\,\eta(W_4(Y,V)W\;)+\eta(V)\,\eta(W_4(U,Y)\,W\\&\eta(W)\,\eta(W_4(U,V)\,Y)-g(Y,U)\,\eta(W_4(\xi,V)\,W)\\&-g(Y,V)\,\eta(W_4(U,\xi)\,W)-g(Y,W)\,\eta(W_4(U,V)\,\xi)]=0.\end{aligned} \tag{4.6}$$

The above equation states that either $f_1=f_3$ or

$$\begin{aligned}&W_4(U,V,W,Y)-\eta(V)\,\eta(W_4(U,V)W)\\&+\eta(U)\,\eta(W_4(Y,V)W)+\eta(V)\,\eta(W_4(U,Y)W)\\&+\eta(W)\,\eta(W_4(U,V)\,Y)-g(Y,U)\,\eta(W_4(\xi,V)\,W)\\&-g(Y,V)\,\eta(W_4(U,\xi)\,W)-g(Y,W)\,\eta(W_4(U,V)\,\xi)=0.\end{aligned} \tag{4.7}$$

If $f_1\neq f_3$, then equation (4.7) must be true. Now we proceed under the assumption that $f_1\neq f_3$. Putting $U=Y$ in (4.7) and using equations (4.1) and (4.2), we get

$$\begin{aligned}&W_4(U,V,W,Y)+\eta(V)\,\eta(W_4(Y,Y)W)\\&+\eta(W)\,\eta(W_4(Y,V)Y)-g(Y,Y)\,\eta(W_4(\xi,V)\,W)\\&-g(Y,W)\,\eta(W_4(Y,V)\,\xi)=0.\end{aligned} \tag{4.8}$$

Let $\{e_1,e_2,\ldots\ldots,e_{2n},e_{2n+1}\xi\}$ is a local orthonormal basis of vector fields in $M\,(f_1,\,f_2,\,f_3)$, Putting $Y=e_i$ in the above equation and taking the summation over $1\le i\le 2n+1$, we get

$$\sum_{i=1}^{2n+1} W_4(e_i, V, W, e_i) + \eta(V)\sum_{i=1}^{2n+1} \eta(W_4(e_i, e_i)W)$$
$$+\eta(W)\sum_{i=1}^{2n+1} \eta(W_4(e_i, V)e_i) - (2n+1)\,\eta(W_4(\xi, V)\,W)$$
$$-\sum_{i=1}^{2n+1} g(e_i, W)\,\eta(W_4(e_i, V)\,\xi) = 0. \quad (4.9)$$

Now using the equations (1.2), (2.1), (2.2), (2.8), (2.9), (3.11) and (4.1), we get

$$S(V, W) = 2n(f_1 - f_3)g(V, W). \quad (4.10)$$

On taking $W = \xi$ in the equation (4.10), we get

$$QV = 2n(f_1 - f_3)V. \quad (4.11)$$

and on taking $V = W = \xi$ in the equation(4.10), we get

$$r = 2n(2n+1)(f_1 - f_3). \quad (4.12)$$

Theorem 4.1 The Ricci-tensor S and the Ricci-operator Q and the scalar curvature r of a W_4 semi-symmetric generalized Sasakian-space-form are given by the equations (4.10), (4.11) and (4.12) respectively.

Acknowledgment

The author is thankful to Professor Ram Nivas, Department of Mathematics and Astronomy, University of Lucknow, providing suggestions for the improvement of this paper.

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