Common Fixed Point Theorem in b-metric Space for compatible mapping of type (A)

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Abstract: *The aim of this paper to prove a unique common fixed point theorem is b-metric space for compatible mapping of type (A) for two self mapping using contraction conditions.*

Keywords: *Fixed point b-metric space, compatible mapping of type A.*

Introduction: Fixed point theory is an important branch of the functional analysis. In 1989 Bakhtin[1] worked on the contraction mapping principle in almost metric space. In 1993 Czerwik [5] extended the result of contraction mapping in b-metric space. Czerwik[5] presented various problems of the convergence of measurable function with respect to measure in b-metric space. Since then, the fixed point theory of the variational principle for single valued and multivalued operators in b-metric space was used by many authors Mehmat Kir [9], Boriceanu [4], M Bota [3] and Pacurar [10] presented generalization of Banach [2] fixed point theorem in b-metric space.
In 1986 G. Jungck [7] defined compatible mapping. In 1993 G Jungck et.al [8] extended compatible mapping by introducing a compatible mapping of type (A).
In this paper, we extended common fixed point theorem in b-metric space for two mapping using compatible mapping of type (A).

Definition 1.1. Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d: X \times X \to R_+$ is called a b- metric provided that for all $x, y, z \in X$

1) $d(x,y) = 0$ if and only if $x = y$,
2) $d(x,y) = d(y,x)$,
3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

A pair (X, d), is called a b-metric space. It is clear that definition of b-metric space is an extension of usual metric space.

Some examples of b-metric spaces are given below:

Example 1.2. By [9] The set $l_p(R)$ (with 0 < p < 1),
where $l_p(R) := \{(x_n) \subset R \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the function $d: l_p(R) X\, l_p(R) \to R$,

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$$

where $x = x_n, y = y_n \in l_p(R)$ is a b-metric space. By an elementary calculation we obtain that $d(x,z) \leq 2^{\frac{1}{p}}[d(x,y) + d(y,z)]$.

Example 1.3. By [9] Let $X = \{0,1,2\}$ and $d(2,0) = d(0,2) = m \geq 2$,
$d(0,1) = d(1,2) = d(1,0) = d(2,1) = 1$
and $d(0,0) = d(1,1) = d(2,2) = 0$.
then
$d(x,y) \leq \frac{m}{2}[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.
if $m > 2$ then the triangle inequality does not hold.

Definition 1.4. Let (X, d) be a b-metric space .Then a sequence $\{x_n\}$ in X is called a

Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 1.6. Let (X, d) be a b-metric space then a sequence $\{x_n\}$ in X is called convergent sequence if and only if there exists $x \in X$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in N$, such that for all $n \geq n(\epsilon)$ we have $d(x_n, x) < \epsilon$. Thus b- metric space is complete if every Cauchy sequence convergent.

Definition 1.7. In 1993, G.Jungck, Y.J.Cho and P.P.Murthy [15] established new concept of compatible mappings i.e. compatible mapping of type (A) .Two self maps S and T of a metric space (X ,d) are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0,$$

and

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some t ∈ X.

THEOREM-2.1 : Let (X, d) be a complete b-metric space with constant $s \geq 1$and S and T are two self mappings such that

(i) $T(X) \subseteq S(X)$,

(ii) One of S or T be continuous,

(iii) $(S, T)$is compatible of type (A),

(iv) $d(Tx,Ty) \leq a\, max\{d(Tx,Sy), d(Sx,Sy), d(Ty,Sy), d(Tx,Sx)\} + b\{d(Ty,Sx)\}$

where $a + 2s \leq 1\ \forall\ x, y \in X$ then S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ .As $T(X) \subseteq S(X)$ then there exist x_{2n+1} and x_{2n} in X such that

$$Tx_{2n} = Sx_{2n+1}, \quad n = 1,2,3 \ldots \ldots$$

Now by equation (iv),

$$d(Sx_{2n+1}, Sx_{2n+2}) = d(Tx_{2n}, Tx_{2n+1}),$$
$$\leq a\ max\ \{ d(Tx_{2n}, Sx_{2n+1}), d(Sx_{2n}, Sx_{2n+1}), d(Tx_{2n+1}, Sx_{2n+1}), d(Tx_{2n}, Sx_{2n})\} + b\{d(Tx_{2n+1}, Sx_{2n})\},$$
$$\leq a\ max\ \{ d(Sx_{2n+1}, Sx_{2n+1}), d(Sx_{2n}, Sx_{2n+1}), d(Sx_{2n+2}, Sx_{2n+1}), d(Sx_{2n+1}, Sx_{2n})\} + b\{d(Sx_{2n+2}, Sx_{2n})\},$$
$$\leq a\ max\ \{d(Sx_{2n}, Sx_{2n+1}), d(Sx_{2n+2}, Sx_{2n+1})\} + b\{d(Sx_{2n+2}, Sx_{2n})\},$$
$$\leq a\ max\ \{d(Sx_{2n}, Sx_{2n+1}), d(Sx_{2n+1}, Sx_{2n+2})\} + sb\{d(Sx_{2n+2}, Sx_{2n+1}) + d(Sx_{2n+1}, Sx_{2n})\},$$

Case –I : If suppose $d(Sx_{2n}, Sx_{2n+1}) > d(Sx_{2n+2}, Sx_{2n+1})$,

$$d(Sx_{2n+1}, Sx_{2n+2}) \leq a\ d(Sx_{2n}, Sx_{2n+1}) + bs\ d(Sx_{2n+2}, Sx_{2n+1}) + bsd(Sx_{2n+1}, Sx_{2n}),$$

$$(1 - sb)d(Sx_{2n+1}, Sx_{2n+2}) \leq (a + sb)\ d(Sx_{2n}, Sx_{2n+1}),$$

$$d(Sx_{2n+1}, Sx_{2n+2}) \leq \frac{(a+sb)}{(1-sb)} d(Sx_{2n}, Sx_{2n+1})$$

$$d(Sx_{2n+1}, Sx_{2n+2}) \leq k_1 d(Sx_{2n}, Sx_{2n+1}),$$

$$k_1 = \frac{(a + sb)}{(1 - sb)} < 1 \ldots \ldots (2.2)$$

Case –I : If suppose $d(Sx_{2n+1}, Sx_{2n+2}) > d(Sx_{2n}, Sx_{2n+1})$,

$$d(Sx_{2n+1}, Sx_{2n+2}) \leq ad\ (Sx_{2n+1}, Sx_{2n+2}) + bs\ d(Sx_{2n+2}, Sx_{2n+1}) + bs\ d(Sx_{2n+1}, Sx_{2n})$$

$$(1 - a - sb)d(Sx_{2n+1}, Sx_{2n+2})$$

$\leq$

$bs\, d(Sx_{2n+1}, Sx_{2n})$

, $d(Sx_{2n+1}, Sx_{2n+2})$

$\leq \frac{bs}{(1-a-sb)} d(Sx_{2n}, Sx_{2n+1})$,

$$k_2 = \frac{bs}{(1-a-sb)} < 1 \ldots \ldots (2.3)$$

By equation (2.2) and (2.3),

Let $k = max\{k_1, k_2\}$,

Therefore $k < 1$,

Since $k_1 < 1$ and $k_2 < 1$,

$d(Sx_{2n+1}, Sx_{2n+2}) \leq k\, d(Sx_{2n}, Sx_{2n+1})$,

$d(Sx_{2n+1}, Sx_{2n+2}) \leq k^2\, d(Sx_{2n-1}, Sx_{2n})$,

$d(Sx_{2n+1}, Sx_{2n+2}) \leq$

$k^3\, d(Sx_{2n-2}, Sx_{2n-1})$

,

:

:$d(Sx_{2n+1}, Sx_{2n+2}) \leq k^{2n}\, d(Sx_0, Sx_1)$.

Now we show that $\{Sx_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $m, n > 0$ with $m > n$.

$d(Sx_n, Sx_m) \leq s\{d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_m)\}$

,

$\leq s\, d(Sx_n, Sx_{n+1}) + s^2 d(Sx_{n+1}, Sx_{n+2}) + s^2 d(Sx_{n+2}, Sx_m)$

,

$\leq s\, d(Sx_n, Sx_{n+1}) + s^2 d(Sx_{n+1}, Sx_{n+2}) + s^3 d(Sx_{n+2}, Sx_{n+3}) + s^3 d(Sx_{n+2}, Sx_{n+3}) \ldots$

,

$\leq$

$sk^n\, d(Sx_0, Sx_1) + s^2 k^{n+1}\, d(Sx_0, Sx_1) + s^3 k^{n+2}\, d(Sx_0, Sx_1) + s^4 k^{n+3}\, d(Sx_0, Sx_1) \ldots$

,

$\leq sk^n\, d(Sx_0, Sx_1)\{1 + sk + (sk)^2 + (sk)^3 + \cdots ..\}$,

$\leq \frac{sk^n}{1-sk}\ d(Sx_0, Sx_1)$,

when we take $m, n \to \infty$, we arrive at

$\lim_{n\to\infty} d(Sx_n, Sx_m) = 0$.

Hence $\{Sx_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

In view of completeness of the space, sequence $\{Sx_n\}_{n=1}^{\infty}$ converges to some u in X.

Since T is a subsequence of S. Therefore it also converges to u.

Since $u \in X$ such that $Tx_{2n} = Sx_{2n} = u$.

Now, we shall prove $Su = u$ then

$d(u, Su) \leq s\{d(u, Tx_{2n}) + d(Tx_{2n}, Su)\}$,

S is continuous and S, T are compatible of type A such that

$STx_{2n} \to Su,\ TSx_{2n} \to Su,\ SSx_{2n} \to Su$

$d(u, Su) \leq$

$s\{d(u, Tx_{2n}) + d(Tx_{2n}, TSx_{2n})\}$,

$\leq s\, d(u, Tx_{2n})$

$+s\, a \max\{d(Tx_{2n}, SSx_{2n})$,

$d(Sx_{2n}, SSx_{2n}), d(TSx_{2n}, SSx_{2n})$,

$d(Tx_{2n}, Sx_{2n})\} + sb\, d(TSx_{2n}, SSx_{2n})$,

Taking as $n \to \infty$

$\leq\ s\, d(u, u) + s\, a \max\{d(u, Su)\ d(u, Su), d(u, Su)$,

$d(Su, Su), d(u, u)\} + sb\, d(Su, Su)$,

$(1 - sa) d(u, Su) \leq 0$,

$d(u, Su) \leq 0$,

This is contradiction. Therefore $u = su$.

Now,

$d(Tu, TSx_{2n}) \leq\ a \max\{d(Tu, SSx_{2n})$,

$(Su, SSx_{2n}), d(TSx_{2n}, SSx_{2n}), d(Tu, Su)\}$

$+\ b\, d(TSx_{2n}, SSx_{2n})$,

S is continuous and S, T are compatible of type A such that

$STx_{2n} = TSx_{2n} = SSx_{2n} = Su = u$

$d(Tu, u) \leq\ a \max\{d(Tu, u)$,

$(u, u), d(u, u), d(Tu, u)\} +\ b\, d(u, u)$,

$(1 - a) d(Tu, u) \leq\ 0$,

$d(Tu, u) \leq\ 0$,

This is contradiction. Therefore $u = Tu$. Hence u is the common fixed point of S and T.

Uniqueness: Let u and v be two common fixed points of S and T, so $u = Su = Tu$ and $v = Sv = Tv$, then we have

$d(u,v) = d(Tu,Tv) \leq a$
$\max\{d(Tu,Sv), d(Su,Sv), d(Tv,Sv), d(Tu,Su)\} + b\,\{d(Tv,Su)\}$ $d(u,v) \leq a$
$\max\{d(u,v), d(u,v), d(v,v), d(u,u)\} + b\,\{d(v,u)\}$
,

$$d(u,v) \leq a$$
$\max\{d(u,v), d(u,v)\} + b\,\{d(u,v)\},$

$$\Rightarrow (1-a-b)d(u,v) \leq 0,$$
$$\Rightarrow d(u,v) = 0, \text{ i.e. } u = v.$$

Hence u is the unique common fixed point of S and T.

Corollary 2.2: Let (X,d) be a complete b-metric space with constant $s \geq 1$and S and T are two self mappings such that

(i) $T(X) \subseteq S(X)$,

(ii) One of S or T be continuous,

(iii) $(S,T)$is compatible of type (A),

(iv) $d(Tx,Ty) \leq a\,\{d(Tx,Sy)\} + b\{d(Sx,Sy)\} + c\{d(Ty,Sx)\}$

where $b + 2cs \leq 1\ \forall\ x,y \in X$ then S and T have a unique common fixed point.

Corollary 2.3: Let (X,d) be a complete b-metric space with constant $s \geq 1$and S and T are two self mappings such that

1) $T(X) \subseteq S(X)$,

2) One of S or T be continuous,

3) $(S,T)$is compatible of type (A),

4) $d(Tx,Ty) \leq a\,max\begin{Bmatrix} d(Tx,Sy), d(Sx,Sy), d(Ty,Sy,) \\ d(Tx,Sx), d(Ty,Sx) \end{Bmatrix}$

Where $a \leq 1$ and $a(1+s) \leq 1$, $\forall\ x,y \in X$ then S and T have a unique common fixed point.

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