SOME PROPERTIES OF S-α ANTI FUZZY SEMIGROUPS

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Abstract

The notion of S-α anti conjugate fuzzy semigroups is introduced and order of an S-α anti fuzzy semigroup is defined. Quotient of S-α anti fuzzy cosets of an S-α anti fuzzy normal subsemigroup is defined and some of its properties are proved. It is also studied about the image and inverse image of S-α anti fuzzy semigroup under S-semigroup anti homomorphism.

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1 *Introduction and Preliminaries*

Zadeh introduced the concept of fuzzy set in 1965 [1]. Fuzzy subgroup was defined by Rosenfeld [2] and it was generalized by Anthony and Sherwood [4]. W.B.Vasantha Kandhasami studied about Smarandache fuzzy semigroups [8]. P.K.Sharma introduced the concept of α-fuzzy set, α-fuzzy group, α-fuzzy coset, α-fuzzy normal subgroup and analyzed about their properties[12]. Gowri.R and Rajeswari.T introduced the concept of S-α anti fuzzy semigroup and S-α anti fuzzy normal subsemigroup and obtained their properties [14]. In this paper order of an S-α anti fuzzy semigroup, S-α anti conjugate fuzzy semigroups, Quotient of S-α anti fuzzy cosets of an S-α anti fuzzy normal subsemigroup are defined and discussed their characterisations.

Definition 1.1 Let X be a non empty set. A **fuzzy subset** A of X is a function $A : X \to [0, 1]$.

Definition 1.2 A fuzzy subset A of a group G is called an **anti fuzzy subgroup** of G if
(i)$A(xy) \leq max\{A(x), A(y)\}$
(ii) $A(x^{-1}) = A(x)$, for all $x, y \in G$.

Definition 1.3 A semigroup S is said to be a **Smarandache-semigroup** **(S-semigroup)** if there exists a proper subset P of S which is a group under the same binary operation in S.

Result 1.4 [*10*] If $A : G \to [0, 1]$ is an anti fuzzy subgroup of a group G, then
(i) $A(x) \geq A(e)$, where e is the identity element of G.
(ii) $A(xy^{-1}) = A(e) \Rightarrow A(x) = A(y)$, for all $x, y \in G$.

Definition 1.5 Let S be an S-semigroup. A fuzzy subset A of S is said to be a **Smarandache anti fuzzy semigroup (S-anti fuzzy semigroup)** if $A : S \to [0, 1]$ is such that A is restricted to atleast one proper subset P of S which is a group and the restriction map $A_P : P \to [0, 1]$ is an anti fuzzy group.

Definition 1.6 Let A be a fuzzy subset of a group G. Let $\alpha \in [0,1]$. Then **α-anti fuzzy subset of G(with respect to a fuzzy set A)**, denoted by A_α, is defined as $A_\alpha(x) = max\{A(x), 1-\alpha\}$, for all $x \in G$.

Definition 1.7 Let A be a fuzzy subset of a group G and $\alpha \in$[0,1]. Then A is called an **α-anti fuzzy subgroup** of G if A_α is an anti fuzzy group.

Definition 1.8 Let f be a mapping from a set G_1 into a set G_2. Let A and B be fuzzy subsets of G_1 and G_2 respectively. Then $f(A)$ and $f^{-1}(B)$ are respectively the image of fuzzy set A and the inverse image of fuzzy set B, defined as

$$f(A)(y) = \begin{cases} Sup\{A(x)/x \in f^{-1}(y)\}, & if\ f^{-1}(y) \neq \Phi \\ 0, & if\ f^{-1}(y) = \Phi, y \in G_2 \end{cases}$$

and $f^{-1}(B)(x) = B(f(x))$, for every $x \in G_1$.

Remark 1.9 [12] (i) Clearly $f(A)(f(x)) \geq A(x)$, for every element $x \in G_1$.
(ii) If f is a bijective map, then $f(A)(f(x)) = A(x)$, for every $x \in G_1$.
(iii) Let $f : X \to Y$ be a mapping and A and B be two fuzzy subsets of X and Y respectively. Then (a) $f^{-1}(B_\alpha) = (f^{-1}(B))_\alpha$ and (b) $f(A_\alpha) = (f(A))_\alpha$.

Definition 1.10 Let G be an S-semigroup. Let A be a fuzzy subset of G and let $\alpha \in$ [0,1]. A is called a **Smarandache - α anti fuzzy semigroup (S-α anti fuzzy semigroup)** if there exists a proper subset P of G which is a group and the restriction of A to $P(A_P : P \to [0,1])$ is such that A_{P_α} is an anti fuzzy group. That is,
(i)$A_{P_\alpha}(xy) \leq max\{A_{P_\alpha}(x), A_{P_\alpha}(y)\}$
(ii) $A_{P_\alpha}(x^{-1}) = A_{P_\alpha}(x)$, for all $x, y \in P$.

Definition 1.11 Let G be an S-semigroup and $\alpha \in [0,1]$. Let A be an S-α anti fuzzy semigroup of G relative to a group P in G and let $x \in P$.
A smarandache-α anti fuzzy right coset (S-α anti fuzzy right coset) of A in G, denoted by $A_{P_\alpha}x$, is defined as
$(A_{P_\alpha}x)(g) = max\{A_P(gx^{-1}), 1-\alpha\}$, for all $g \in P$.

Definition 1.12 Let G be an S-semigroup and $\alpha \in [0,1]$. Let A be an S-α anti fuzzy semigroup of G relative to a group P in G and let $x \in P$.
A smarandache-α anti fuzzy left coset (S-α anti fuzzy left coset) of A in G, denoted by xA_{P_α}, is defined as $(xA_{P_\alpha})(g) = max\{A_P(x^{-1}g), 1-\alpha\}$, for all $g \in P$.

Definition 1.13 Let G be an S-semigroup and $\alpha \in [0,1]$. An S-α anti fuzzy semigroup A of G relative to a group P in G is said to be a **Smarandache-α anti fuzzy normal subsemigroup (S-α anti fuzzy normal subsemigroup)** of G if $xA_{P_\alpha} = A_{P_\alpha}x$, for all $x \in P$.

Result 1.14 [14] Let A be an S-α anti fuzzy normal subsemigroup of an S-semigroup G relative to a group P in G. Define a set
$G_{A_{P_\alpha}} = \{x \in P / A_{P_\alpha}(x) = A_{P_\alpha}(e)$, e is the identity of P$\}$. Then $G_{A_{P_\alpha}}$ is a normal subgroup of P.

Result 1.15 [14] Let A be an S-α anti fuzzy normal subsemigroup of an S-semigroup G relative to a group P in G. Then for all $x, y \in P$
(i)$xA_{P_\alpha} = yA_{P_\alpha} \Leftrightarrow x^{-1}y \in G_{A_{P_\alpha}}$
(ii)$A_{P_\alpha}x = A_{P_\alpha}y \Leftrightarrow xy^{-1} \in G_{A_{P_\alpha}}$.

Definition 1.16 Let S and $S^{'}$ be any two S-semigroups. A map $\phi : S \to S^{'}$ is said to be an **S-semigroup homomorphism** if ϕ is restricted to subgroups $A \subset S$ and $A^{'} \subset S^{'}$ such that $g : A \to A^{'}$, which is defined as $g(x) = \phi(x)$, $x \in A$, is a group homomorphism and ϕ is said to be an S-semigroup isomorphism if g is also bijective.

Result 1.17 [14] Let A be an S-α anti fuzzy semigroup of an S-semigroup G relative to a group P in G. Then the following conditions are equivalent for all $x, y \in P$
(i)A is an S-α anti fuzzy normal subsemigroup.
(ii)$A_{P_\alpha}(xyx^{-1}) \geq A_{P_\alpha}(y)$
(iii)$A_{P_\alpha}(xyx^{-1}) = A_{P_\alpha}(y)$
(iv)$A_{P_\alpha}(xy) = A_{P_\alpha}(yx)$
(v)$xA_{P_\alpha}x^{-1} = A_{P_\alpha}$
(vi)$A_{P_\alpha}(y^{-1}xy) = A_{P_\alpha}(x)$.

2 *Properties of S-α anti fuzzy semigroup*

In this section S-α anti conjugate fuzzy semigroups, quotient of S-α anti fuzzy cosets of an S-α anti fuzzy normal subsemigroup are defined and their properties are analyzed. Throughout this section α will always denote a member of $[0, 1]$.

Definition 2.1 Let A be an S-α anti fuzzy semigroup of an S-semigroup G relative to a group P in G. Define $H = \{x \in P/A_{P_\alpha}(x) = A_{P_\alpha}(e),\ e$ is the identity of $P\}$. Then the order of A, denoted by $O(A)$, is defined as $O(A) = O(H)$.

Example 2.2 Consider $S(3)$ which is an S-semigroup. Let $A : S(3) \to [0, 1]$ be defined as

$$A(x) = \begin{cases} 0.45, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.5, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ 0.8, & \text{otherwise} \end{cases}$$

Let $P = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$. and $\alpha = 0.4$.
Then A is an S-α anti fuzzy semigroup of $S(3)$ relative to P.
If $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in P$ then $A_{P_\alpha}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = 0.6 = A_{P_\alpha}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.
Therefore, if $H = \{x \in P/A_{P_\alpha}(x) = A_{P_\alpha}(e),\ e$ is the identity element of $P\}$,
then $H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$
Therefore $O(H) = 2$ and hence $O(A) = 2$.

Definition 2.3 Let A and B be two S-α anti fuzzy semigroups of an S-semigroup G relative to the same group P in G. A and B are said to be **Smarandache-α anti conjugate fuzzy semigroups (S-α anti conjugate fuzzy semigroups)** of G relative to P if there exists $g \in P$ such that $A_{P_\alpha}(x) = B_{P_\alpha}(g^{-1}xg)$, for all $x \in P$.

Example 2.4 Consider $S(3)$ which is an S-semigroup. Let $A : S(3) \to [0, 1]$ be defined as

$$A(x) = \begin{cases} 0.45, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.65, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} and \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ 0.75, & \text{otherwise} \end{cases}$$

and $B : S(3) \to [0, 1]$ be defined as

$$B(x) = \begin{cases} 0.45, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.65, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} and \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ 0.85, & \text{otherwise} \end{cases}$$

Let $P = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$ and $\alpha = 0.6$.

Then A and B are S-α anti fuzzy semigroups. Let $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in P$.
If $x = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in P$ then $A_{P_\alpha}(x) = 0.65 = B_{P_\alpha}(g^{-1}xg)$.
Similarly $A_{P_\alpha}(x) = B_{P_\alpha}(g^{-1}xg)$, for all $x \in P$.
Therefore A and B are S-α anti conjugate fuzzy semigroups of $S(3)$ ralative to P.

Remark 2.5 If A and B are two S-α anti fuzzy semigroups of an S-semigroup G relative to the same group P in G, then it is easy to see that $A_{P_\alpha}(gxg^{-1}) = B_{P_\alpha}(x) \Leftrightarrow A_{P_\alpha}(x) = B_{P_\alpha}(g^{-1}xg)$, for all $x \in P$. Therefore A and B are S-α anti conjugate fuzzy semigroups of G relative to P iff $A_{P_\alpha}(gxg^{-1}) = B_{P_\alpha}(x)$.

Theorem 2.6 If A and B are S-α anti conjugate fuzzy semigroups of an S-semigroup G relative to a group P in G, then $O(A) = O(B)$.

Proof: Since A and B are S-α anti conjugate fuzzy semigroups of G relative to P, there exists $g \in G$ such that $A_{P_\alpha}(x) = B_{P_\alpha}(g^{-1}xg)$, for all $x \in P$. We define $H = \{x \in P/A_{P_\alpha}(x) = A_{P_\alpha}(e)\}$ and $K = \{x \in P/B_{P_\alpha}(x) = B_{P_\alpha}(e)\}$, where e is the identity element of P. If $x, y \in H$ then $A_{P_\alpha}(xy) \leq max\{A_{P_\alpha}(x), A_{P_\alpha}(y)\} = A_{P_\alpha}(e)$ and hence $A_{P_\alpha}(xy) = A_{P_\alpha}(e)$.
Therefore $xy \in H$. If $x \in H$ then $x^{-1} \in P$. Now $A_{P_\alpha}(x^{-1}) = A_{P_\alpha}(x) = A_{P_\alpha}(e)$ which implies $x^{-1} \in H$. Therefore H is a subgroup of P. Similarly K is also a subgroup of P. To prove $O(A) = O(B)$, by the definition of order, it is enough to prove that $O(H) = O(K)$. Let x be an arbitrary element in H. Therefore there exists $g \in P$ such that $B_{P_\alpha}(g^{-1}xg) = A_{P_\alpha}(x) = B_{P_\alpha}(g^{-1}eg) = B_{P_\alpha}(e)$. This implies $g^{-1}xg \in K$ and hence $x \in gKg^{-1}$. Therefore $H \subset gKg^{-1}$. Now let $x \in K$. By assumption there exists the same $g \in P$ such that $A_{P_\alpha}(gxg^{-1}) = B_{P_\alpha}(x) = A_{P_\alpha}(g^{-1}eg) = A_{P_\alpha}(e)$. Therefore $gxg^{-1} \in H$ which implies $x \in g^{-1}Hg$. Thus $K \subset g^{-1}Hg \Rightarrow gK \subset Hg$. Also $H \subset gKg^{-1} \Rightarrow Hg \subset gK$. Therefore $Hg = gK$ which implies $H = gKg^{-1}$ and hence $O(H) = O(gKg^{-1})$. Since K is a subgroup of P contained in G, we have $O(xKx^{-1}) = O(K)$, for all $x \in P$. Therefore $O(K) = O(gKg^{-1}) = O(H)$. Hence the theorem. □

Theorem 2.7 Let A be an S-α anti fuzzy normal subsemigroup of an S-semigroup G relative to a group P in G. Let G/A_{P_α} denote the collection of all S-α anti fuzzy cosets of A in G relative to P. If $A_{P_\alpha}x \bigotimes A_{P_\alpha}y = A_{P_\alpha}xy$, $x, y \in P$ then $\bigotimes$ is a well defined binary operation on G/A_{P_α}.

Proof: Since A is an S-α anti fuzzy normal subsemigroup of G relative to P,
$xA_{P_\alpha} = A_{P_\alpha}x$, for all $x \in P$. Also $G/A_{P_\alpha} = \{A_{P_\alpha}x/x \in P\}$.
Let $A_{P_\alpha}x = A_{P_\alpha}x'$ and $A_{P_\alpha}y = A_{P_\alpha}y'$
where $x, y, x', y' \in P$. If $g \in P$ then

$$
\begin{aligned}
(A_{P_\alpha}xy)(g) &= max\{A_P(g(xy)^{-1}), 1-\alpha\} \\
&= max\{A_P((gy^{-1})x^{-1}), 1-\alpha\} \\
&= (A_{P_\alpha}x)(gy^{-1}) \\
&= (A_{P_\alpha}x')(gy^{-1}) \\
&= (x'A_{P_\alpha})(gy^{-1}) \\
&= max\{A_P((x'^{-1}g)y^{-1}), 1-\alpha\} \\
&= (A_{P_\alpha}y)(x'^{-1}g) = (y'A_{P_\alpha})(x'^{-1}g) \\
&= max\{A_P((y'^{-1}x'^{-1})g), 1-\alpha\} \\
&= (x'y'A_{P_\alpha})(g) = (A_{P_\alpha}x'y')(g).
\end{aligned}
$$

$$\text{Hence } A_{P_\alpha}xy = A_{P_\alpha}x^{'}y^{'}.$$

Therefore $\bigotimes$ is a well defined binary operation on G/A_{P_α}. □

Theorem 2.8 The set G/A_{P_α} of all S-α anti fuzzy cosets of an S-α anti fuzzy normal subsemigroup A of an S-semigroup G relative to a group P in G is a group under the binary operation $\bigotimes$ defined in theorem 2.7.

Proof: By using the binary operation $\bigotimes$, it can be easily proved that the identity element of G/A_{P_α} is $A_{P_\alpha}e$ where e is the identity element of the group P and the inverse of an element $A_{P_\alpha}x$ in G/A_{P_α} is $A_{P_\alpha}x^{-1}$ where $x \in P$. □

Definition 2.9 Let A be an S-α anti fuzzy normal subsemigroup of an S-semigroup G relative to a group P in G. Then the set G/A_{P_α} of all S-α anti fuzzy cosets of A in G relative to P is a group and is called a factor group or quotient group of G by A_{P_α}.

Example 2.10 Consider $G = S(3)$ which is an S-semigroup.
Let $A : S(3) \to [0, 1]$ be defined as

$$A(x) = \begin{cases} 0.4, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.5, & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} and \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ 0.6, & \text{otherwise} \end{cases}$$

Let $P = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$ and $\alpha = 0.6$.

For $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, an S-α anti fuzzy right coset of A in G is given by

$$(A_{P_\alpha}x)(g) = \begin{cases} 0.5, & if\ g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.4, & if\ g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ 0.5, & if\ g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \end{cases}$$

Clearly A is an S-α anti fuzzy normal subsemigroup of G and $G/A_{P_\alpha} = \{A_{P_\alpha}x/x \in P\}$. It can be easily verified that G/A_{P_α} is a group under the binary operation $\bigotimes$ defined in theorem 2.7.

Theorem 2.11 Let G be an S-semigroup and A be an S-α anti fuzzy normal subsemigroup of G relative to a group P in G. Then the mapping
$f : P \to G/A_{P_\alpha}$ defined by $f(x) = A_{P_\alpha}x$ is an onto homomorphism with
$kerf = G_{A_{P_\alpha}}$.

Proof: Let $x, y \in P$. Then $f(xy) = A_{P_\alpha}xy = (A_{P_\alpha}x) \bigotimes (A_{P_\alpha}y) = f(x) \bigotimes f(y)$ and hence f is a homomorphism. By definition of f, it is obvious that f is surjective. Now $Kerf = \{x \in P/f(x) = A_{P_\alpha}e\} = \{x \in P/xe^{-1} \in G_{A_{P_\alpha}}\}$[by theorem 1.15]$= \{x \in P/x \in G_{A_{P_\alpha}}\} = G_{A_{P_\alpha}}$.
□

Theorem 2.12 Let A be an S-α anti fuzzy normal subsemigroup of an S-semigroup G relative to a group P in G. Then the group G/A_{P_α} is isomorphic to the quotient group $P/G_{A_{P_\alpha}}$.

Proof: We know that $G/A_{P_\alpha} = \{A_{P_\alpha}x/x \in P\}$ and
$G_{A_{P_\alpha}} = \{x \in P/A_{P_\alpha}(x) = A_{P_\alpha}(e)$, e is the identity of $P\}$. Since $G_{A_{P_\alpha}}$ is a normal subgroup of P [by theorem 1.14], $P/G_{A_{P_\alpha}}$ is a quotient group.

Let $\psi : G/A_{P_\alpha} \to P/G_{A_{P_\alpha}}$ be defined as $\psi(A_{P_\alpha}x) = (G_{A_{P_\alpha}})x, x \in P$. If $x, y \in P$, then $\psi(A_{P_\alpha}x \bigotimes A_{P_\alpha}y) = \psi(A_{P_\alpha}xy) = (G_{A_{P_\alpha}})xy = (G_{A_{P_\alpha}}x)(G_{A_{P_\alpha}}y)$
$= \psi(A_{P_\alpha}x)\psi(A_{P_\alpha}y)$ and hence ψ is a homomorphism. Now, if $\psi(A_{P_\alpha}x) = \psi(A_{P_\alpha}y)$ then $xy^{-1} \in G_{A_{P_\alpha}}$ which implies that $A_{P_\alpha}x = A_{P_\alpha}y$ [by theorem 1.15]. Therefore ψ is injective. By the definition of ψ, it is easy to see that ψ is onto. Therefore G/A_{P_α} is isomorphic to $P/G_{A_{P_\alpha}}$. □

Remark 2.13 If A is an S-α anti fuzzy semigroup of an S-semigroup G relative to a group P in G, then clearly $A_{P_\alpha}(x) = A_\alpha(x)$, for all $x \in P$.

Theorem 2.14 Let G_1 and G_2 be two S-semigroups and let $f : G_1 \to G_2$ be an S-semigroup homomorphism. If B is S-α anti fuzzy semigroup of G_2 relative to the restricted group in G_2 with respect to f, then $f^{-1}(B)$ is also S-α anti fuzzy semigroup of G_1.

Proof: Since $f : G_1 \to G_2$ is an S-semigroup homomorphism, f is restricted to subgroups $P \subset G_1$ and $Q \subset G_2$ such that $\phi : P \to Q$, which is defined as $\phi(x) = f(x), x \in P$, is a group homomorphism. By assumption, B is an S-α anti fuzzy semigroup of G_2 relative to Q. Therefore B_{Q_α} is an anti fuzzy group. By definition of inverse image, $f^{-1}(B)(x) = B(f(x)), x \in G_1$. Let $x, y \in P$.

$$\begin{aligned}
(i)(f^{-1}(B))_{P_\alpha}(xy) &= (f^{-1}(B))_\alpha(xy)[\text{by remark 2.13}] \\
&= (f^{-1}(B_\alpha))(xy)[\text{by remark 1.9 }] \\
&= B_\alpha(f(xy)) \\
&= B_\alpha(\phi(xy)) \\
&= B_\alpha(\phi(x)\phi(y)) \\
&= B_{Q_\alpha}(\phi(x)\phi(y)) \\
&\leq max\{B_{Q_\alpha}(\phi(x)), B_{Q_\alpha}(\phi(y))\} \\
&= max\{B_\alpha(f(x)), B_\alpha(f(y))\} \\
&= max\{(f^{-1}(B))_\alpha(x), (f^{-1}(B))_\alpha(y)\} \\
Therefore \quad (f^{-1}(B))_{P_\alpha}(xy) &\leq max\{(f^{-1}(B))_{P_\alpha}(x), (f^{-1}(B))_{P_\alpha}(y)\} \\
(ii)(f^{-1}(B))_{P_\alpha}(x^{-1}) &= f^{-1}(B_\alpha)(x^{-1}) \\
&= B_\alpha(f(x^{-1})) \\
&= B_\alpha(\phi(x)^{-1}) \\
&= B_{Q_\alpha}(\phi(x)^{-1}) \\
&= B_{Q_\alpha}(\phi(x)) \\
&= B_\alpha(f(x)) \\
&= (f^{-1}(B))_\alpha(x) \\
Therefore \quad (f^{-1}(B))_{P_\alpha}(x^{-1}) &= (f^{-1}(B))_{P_\alpha}(x).
\end{aligned}$$

By (i) and (ii) $f^{-1}(B)$ is an S-α anti fuzzy semigroup of G_1 relative to P. □

Theorem 2.15 Let G_1 and G_2 be two S-semigroups and let $f : G_1 \to G_2$ be an S-semigroup homomorphism. If B is an S-α anti fuzzy normal subsemigroup of G_2 relative to the restricted group in G_2 with respect to f, then $f^{-1}(B)$ is S-α anti fuzzy normal subsemigroup of G_1.

Proof: As in theorem 2.14, $\phi : P \to Q$,which is defined as $\phi(x) = f(x)$, $x \in P$, is a group homomorphism. By assumption B is an S-α anti fuzzy normal subsemigroup of G_2 relative to Q. Therefore by result [1.17] $B_{Q_\alpha}(xy) = B_{Q_\alpha}(yx), x, y \in Q$.

Let $x, y \in P$. Now $(f^{-1}(B))_{P_\alpha}(xy) = f^{-1}(B_\alpha)(xy) = B_\alpha(f(xy)) = B_\alpha(\phi(xy))$
$= B_{Q_\alpha}(\phi(x)\phi(y)) = B_{Q_\alpha}(\phi(y)\phi(x)) = B_{Q_\alpha}(\phi(yx)) = B_{Q_\alpha}(f(yx)) = B_\alpha(f(yx))$.
Hence $(f^{-1}(B))_{P_\alpha}(xy) = (f^{-1}(B))_{P_\alpha}(yx)$.
Therefore $f^{-1}(B)$ is an S-α anti fuzzy normal subsemigroup of G_1. □

Theorem 2.16 Let G_1 and G_2 be two S-semigroups and let $f : G_1 \to G_2$ be an S-semigroup isomorphism. If A is an S-α anti fuzzy semigroup of G_1 relative to the restricted group in G_1 with respect to f, then $f(A)$ is an S-α anti fuzzy semigroup of G_2.

Proof: Since $f : G_1 \to G_2$ be an S-semigroup isomorphism, f is restricted to subgroups $P \subset G_1$ and $Q \subset G_2$ such that $\phi : P \to Q$, which is defined as
$\phi(x) = f(x), x \in P$, is a group isomorphism. By assumption A is an S-α anti fuzzy semigroup of G_1 relative to P. Therefore A_{P_α} is an anti fuzzy group. By definition of image,

$$f(A)(y) = \begin{cases} Sup\{A(x)/x \in f^{-1}(y)\}, & if\ f^{-1}(y) \neq \Phi \\ 0, & if\ f^{-1}(y) = \Phi, y \in G_2 \end{cases}$$

Let $y_1, y_2 \in Q$. Since ϕ is bijective there exist unique elements $x_1, x_2 \in P$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$.

$$\begin{aligned} (i)(f(A))_{Q_\alpha}(y_1y_2) &= (f(A))_\alpha(y_1y_2) \\ &= max\{f(A)(\phi(x_1)\phi(x_2), 1-\alpha\} \\ &= max\{f(A)(f(x_1x_2)), 1-\alpha\} \\ &= max\{A(x_1x_2), 1-\alpha\} \\ &= A_{P_\alpha}(x_1x_2) \\ &\leq max\{A_{P_\alpha}(x_1), A_{P_\alpha}(x_2)\} \\ &= max\{sup\{A_\alpha(x_1)/f(x_1) = y_1\}, \\ &\quad sup\{A_\alpha(x_2)/f(x_2) = y_2\}\} \\ &= max\{f(A_\alpha)(y_1), f(A_\alpha)(y_2)\}. \end{aligned}$$

Thus $(f(A))_{Q_\alpha}(y_1y_2) \leq max\{(f(A))_{Q_\alpha}(y_1), (f(A))_{Q_\alpha}(y_2)\}$.
(ii) Since ϕ is a homomorphism, $\phi(x_1) = y_1 \Rightarrow \phi(x_1^{-1}) = y_1^{-1}$.
Now, $(f(A))_{Q_\alpha}(y_1^{-1}) = (f(A))_\alpha(y_1^{-1}) = max\{f(A)(y_1^{-1}), 1-\alpha\}$
$= max\{f(A)\phi(x_1^{-1}), 1-\alpha\} = max\{f(A)f(x_1^{-1}), 1-\alpha\}$
$= max\{A(x_1^{-1}), 1-\alpha\} = A_{P_\alpha}(x_1^{-1}) = A_{P_\alpha}(x_1) = sup\{A_{P_\alpha}(x_1)/f(x_1) = y_1\} = (f(A))_{Q_\alpha}(y_1)$.
By (i) and (ii) $f(A)$ is an S-α anti fuzzy semigroup of G_2. □

Theorem 2.17 Let G_1 and G_2 be two S-semigroups and let $f : G_1 \to G_2$ be
S-semigroup isomorphism. If A is an S-α anti fuzzy normal subsemigroup of G_1 relative to the restricted group in G_1 with respect to f, then $f(A)$ is an S-α anti fuzzy normal subsemigroup of G_2.

Proof: As in theorem 2.16 $\phi : P \to Q$, which is defined as $\phi(x) = f(x), x \in P$, is an isomorphism and $A_{P_\alpha}(x_1x_2) = A_{P_\alpha}(x_2x_1)$, for all $x_1, x_2 \in P$ by result 1.17. Let $y_1, y_2 \in Q$. Since ϕ is bijective there exist unique elements $x_2, x_2 \in P$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Now $(f(A))_{Q_\alpha}(y_1y_2) = max\{f(A)(\phi(x_1)\phi(x_2)), 1-\alpha\} = max\{f(A)f(x_1x_2), 1-\alpha\} = max\{A(x_1x_2), 1-\alpha\} = A_{P_\alpha}(x_1x_2)$ which leads to $(f(A))_{Q\alpha}(y_1y_2) = (f(A))_{Q_\alpha}(y_2y_1)$.
Thus $f(A)$ is an S-α anti fuzzy normal subsemigroup of G_2. □

References

[1] Zadeh L. A., "Fuzzy sets", *Information and Control*, 8, 338-353, 1965.

[2] Rosenfeld A., "Fuzzy groups", *Journal of Mathematical analysis and application*, 35, 512-517, 1971.

[3] Anthony J. M. and Sherwood H., "Fuzzy groups redefined", *J. Math. Anal. Appl.*, 69, 124-130, 1979.

[4] Anthony J. M. and Sherwood H., "A characterization of fuzzy subgroups",*Fuzzy Sets and Systems*, 7, 297-305, 1987.

[5] Mashour A. S., Ghanim M. H., Sidky F.I., "Normal Fuzzy Subgroups", Ser. Math., 20(2), 53-59, 1990.

[6] Malik D. S., Mordeson J. N. and Nair P. S., "Fuzzy Normal Subgroups in Fuzzy groups", *J. korean Math. Soc.*, 29(1), 1-8, 1992.

[7] Rajeshkumar, "Fuzzy Algebra", *University of Delhi Publication Division*, 1993.

[8] Vasantha Kandhasamy W. B., "Smarandache Fuzzy Algebra", *American Research Press*, Rehoboth, 2003.

[9] Liu Y. L., "Quotient groups induced by fuzzy subgroups", *Quasigroups and Related Systems*, 11, 71-78, 2004.

[10] Sharma P. K., "α-Anti Fuzzy subgroups", *International Review of Fuzzy Mathematics*, 7(2), 47-58, 2012.

[11] Massa'deh M. O., "On Fuzzy subgroups with operators", *Asian Journal of Mathematics and Statistics*, 5(4), 163-166, 2012.

[12] Sharma P. K., "α-Fuzzy subgroups", *International Journal of fuzzy Mathematics and systems*, 3(1), 47-59, 2013.

[13] Gowri. R and Rajeswari. T, "S-α fuzzy semigroups", *International J. of Math. Sci. and Engg. Appls.*, Vol.9 No.1, 307-318, March 2015,.

[14] Gowri. R and Rajeswari. T, "S-α anti fuzzy semigroups", *International Journal of Engineering Science and Innovative Technology*, Volume 4, Issue 6, 108-118, Nov 2015.