Some Results in the Ring $R_{2p^nq^m} = GF(l)[x]/(x^{2p^nq^m} - 1)$

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Abstract

Let p, q and l be distinct odd primes and $R_{2p^nq^m} = GF(l)[x]/(x^{2p^nq^m} - 1)$.*If* $o(l)_{p^n} = \phi(p^n)/2$, *(n ≥ 1) and* $o(l)_{q^m} = \phi(q^m)/2, (m \geq 1)$ *with gcd (* $\phi(p^n)/2$, $\phi(q^m)/2$*) =1, then the explicit expressions for the complete set of 8mn+4n+4m+2 primitive idempotents in the ring* $R_{2p^nq^m} = GF(l)[x]/(x^{2p^nq^m} - 1)$ *are obtained.*

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1. Introduction

Let F = GF(l) be a field of odd prime order l.Let $\eta \geq 1$ be an integer with gcd(l, η)=1.Let $R_\eta = GF(l)[x]/(x^\eta - 1)$.The minimal cyclic codes of length η over GF(l) are the ideals of the ring R_η generated by the primitive idempotents.For $\eta = 2$, 4, p^n, $2p^n$, p an odd prime and l *is* primitive root mod(η) the primitive idempotent in R_η have been obtained by Arora and Pruthi [1,2] .When $m = p^n q$ where p, q are distinct odd primes and l is a primitive root mod p^n and q both with gcd ($\phi(p^n)/2$, $\phi(q)/2$) = 1, the primitive idempotent in R_η have been obtained by, G.K.Bakshi and Madhu Raka [4]. In this paper, we consider the case when $\eta = 2p^nq^m$ where p, q are distinct odd primes $o(l)_{p^n} = \phi(p^n)/2$, (n ≥ 1) and $o(l)_{q^m} = \phi(q^m)/2, (m \geq 0)$ with gcd ($\phi(p^n)/2, \phi(q^m)/2$)=1.We obtain explicit expressions for all the 8mn+4n+4m+2 primitive idempotents in $R_{2p^nq^m}$ (see theorem 2.5).

2. Primitive Idempotents in $R_{2p^nq^m} = GF(l)[x]/(x^{2p^nq^m} - 1)$

2.1. For $0 \leq s \leq \eta - 1$, let $C_s = \{s, sl, sl^2, ..., sl^{t_s - 1}\}$, where t_s is the least positive integer such that $sl^{t_s} \equiv s \pmod{\eta}$ be the cyclotomic coset containing s, if α denotes a primitive $\eta\, th$ root of unity in some extension field of GF(l) then the polynomial $M^s(x) = \prod_{i \in C_s}(x - \alpha^i)$ is the minimal polynomial of α^s over GF(l).Let M_s be the minimal ideal in R_η generated by $\frac{x^\eta - 1}{M^s(x)}$ and θ_s be the primitive idempotent of M_s then we know by (Theorem1, [4]) the primitive idempotent θ_s corresponding to the cyclotomic coset C_s containing s in $R_{2p^nq^m}$ is given by $\theta_s = \sum_{i=0}^{2p^nq^m - 1} \varepsilon_i x^i$, where $\varepsilon_i = \frac{1}{2p^nq^m} \sum_{j \in C_s} \alpha^{-ij}$ $\forall i \geq 0$.Thus to describe θ_s it becomes necessary to compute ε_i.To compute ε_i numerically, we consider the case when $-C_1 = C_{ab}$ and we get that $\varepsilon_i = \frac{1}{2p^nq^m} \sum_{j \in C_s} \alpha^{-ij} = \frac{1}{2p^nq^m} \sum_{j \in C_{sab}} \alpha^{ij}$ $\forall i \geq 0$ where a, b are defined in lemma2.2.

Lemma2.1. Let p,q,l be distinct odd primes and n ≥ 1, m ≥ 1 are integers, $o(l)_{2p^n} = \frac{\phi(2p^n)}{2}$, $o(l)_{q^m} = \frac{\phi(q^m)}{2}$ and gcd $(\frac{\phi(2p^n)}{2}, \frac{\phi(q^m)}{2})$ = 1.Then $o(l)_{2p^{n-j}q^{m-k}} = \frac{\phi(2p^{n-j}q^{m-k})}{4}$, for all j, k, $0 \le j \le n-1,\ 0 \le k \le m-1$.

Proof. Similar to Lemma2.1 [5].

Lemma2.2. For given distinct odd primes p, q and l, there exists always fixed integers a and b satisfying (a, 2pq$l$) = 1, 1 < a < 2pq, a $\not\equiv l^t$ (mod 2 pq), b $\not\equiv$ r l^t(mod 2pq) , r = a or 1 for $0 \le t \le \frac{\phi(2pq)}{4} - 1$.Further, for 0 ≤ j ≤ n-1 and 0 ≤ k ≤ m-1, the set

$\{1, l, l^2, \ldots, l^{\frac{\phi(2p^{n-j}q^{m-k})}{4}-1}$, a, a$l$, …, a$l^{\frac{\phi(2p^{n-j}q^{m-k})}{4}-1}$,b,b$l$, b$l^2$, …,b $l^{\frac{\phi(2p^{n-j}q^{m-k})}{4}-1}$, ab,ab$l$, …, ab$l^{\frac{\phi(2p^{n-j}q^{m-k})}{4}-1}\}$forms a reduced residue system (mod $2p^{n-j}q^{m-k}$).

Proof. Similar to Lemma2.2 [5].

Remark2.3. On the similar lines we can prove that, for 0 ≤ i ≤ n-1 the set $\{1, l, l^2, \ldots, l^{\frac{\phi(2p^{n-i})}{2}-1}$, b, b$l$, …, b$l^{\frac{\phi(2p^{n-i})}{2}-1}\}$forms a reduced residue system modulo $p^{n-i}$and for 0 ≤ j ≤ m-1, the set $\{1, l, l^2, \ldots, l^{\frac{\phi(2p^{n-j})}{2}-1}$,a, a$l$, …a$l^{\frac{\phi(2p^{n-j})}{2}-1}\}$forms a reduced residue system modulo q^{m-j} .

Note (i): Since in Theorem2.2 we have a $\not\equiv l^k$ (mod 2pq), and b $\not\equiv$ r l^k (mod 2pq), where $0 \le k \le \frac{\phi(2pq)}{4} - 1$ and r = a or 1.Then mainly the following cases holds: For $0 \le s \le \frac{\phi(2pq)}{4} - 1$ we have **Case (i)** a ≡ l^s mod(2p) , b≡ l^s mod(q) , b $\not\equiv l^s$ mod (2p) and a $\not\equiv l^s$ mod (q).

Case (ii) a≡ l^s mod(q) , b≡ l^s mod(2p) , b $\not\equiv l^s$ mod (q) and a $\not\equiv l^s$ mod (2p).

Case(iii)a $\not\equiv l^s$ mod (2p) , b≡ l^s mod(q) , b $\not\equiv l^s$ mod (2p) and a $\not\equiv l^s$ mod(q).

Case(iv)a $\not\equiv l^s$ mod (2p) , a≡ l^s mod(q) , b $\not\equiv l^s$ mod (2p) and b $\not\equiv l^s$ mod(q).

Here we are considering the case (i) because the remaining cases follows in a similar way.

Note (ii): 2 $C_2 = C_2$ or 2 $C_2 = C_{2a}$ or 2 $C_2 = C_{2b}$ or 2 $C_2 = C_{2ab}$.

Similarly q $C_q = C_q$ or q $C_q = C_{bq}$.But here we are considering the case when $2C_2 = C_2$ and q$C_q = C_q$ .The remaining cases follows in similar way.

Theorem2.4. If η = $2p^nq^m$ (n ≥ 1, m ≥ 0), then the 8mn+4n+4m+2 cyclotomic cosets modulo $2p^nq^m$ are given by (i) C_0 = {0}, (ii) $C_{p^nq^m}$ = $\{p^nq^m\}$

For 0 ≤ j ≤ m–1, (iii) $C_{p^nq^j} = \{p^n q^j, p^n q^j l, \ldots, p^n q^j l^{\frac{\phi(q^{m-j})}{2}-1}\}$.

(iv) $C_{2p^nq^j} = \{2p^n q^j, 2p^n q^j l, \ldots, 2p^n q^j l^{\frac{\phi(q^{m-j})}{2}-1}\}$.

(v) $C_{ap^nq^j} = \{ap^n q^j, ap^n q^j l, \ldots, ap^n q^j l^{\frac{\phi(q^{m-j})}{2}-1}\}$.

(vi) $C_{2ap^nq^j} = \{2ap^n q^j, 2ap^n q^j l, \ldots, 2ap^n q^j l^{\frac{\phi(q^{m-j})}{2}-1}\}$.

For $0 \le i \le n-1$, (vii) $C_{p^iq^m} = \{p^iq^m, p^iq^m l, \ldots, p^iq^m l^{\frac{\phi(2p^{n-i})}{2}-1}\}$.

(viii) $C_{2p^iq^m} = \{2p^iq^m, 2p^iq^m l, \ldots, 2p^iq^m l^{\frac{\phi(p^{n-i})}{2}-1}\}$.

(ix) $C_{bp^iq^m} = \{bp^iq^m, bp^iq^m l, \ldots, bp^iq^m l^{\frac{\phi(2p^{n-i})}{2}-1}\}$.

(x) $C_{2bp^iq^m} = \{2bp^iq^m, 2bp^iq^m l, \ldots, 2bp^iq^m l^{\frac{\phi(p^{n-i})}{2}-1}\}$. For $0 \le i \le n-1$, $0 \le j \le m-1$

(xi) $C_{p^iq^j} = \{p^iq^j, p^iq^j l, \ldots, p^iq^j l^{\frac{\phi(2p^{n-i}q^{m-j})}{4}-1}\}$.

(xii) $C_{2p^iq^j} = \{2p^iq^j, 2p^iq^j l, \ldots, 2p^iq^j l^{\frac{\phi(p^{n-i}q^{m-j})}{4}-1}\}$.

(xiii) $C_{ap^iq^j} = \{ap^iq^j, ap^iq^j l, \ldots, ap^iq^j l^{\frac{\phi(2p^{n-i}q^{m-j})}{4}-1}\}$.

(xiv) $C_{2ap^iq^j} = \{2ap^iq^j, 2ap^iq^j l, \ldots, 2ap^iq^j l^{\frac{\phi(p^{n-i}q^{m-j})}{4}-1}\}$.

(xv) $C_{bp^iq^j} = \{bp^iq^j, bp^iq^j l, \ldots, bp^iq^j l^{\frac{\phi(2p^{n-i}q^{m-j})}{4}-1}\}$.

(xvi) $C_{2bp^iq^j} = \{2bp^iq^j, 2bp^iq^j l, \ldots, 2bp^iq^j l^{\frac{\phi(p^{n-i}q^{m-j})}{4}-1}\}$.

(xvii) $C_{abp^iq^j} = \{abp^iq^j, abp^iq^j l, \ldots, abp^iq^j l^{\frac{\phi(2p^{n-i}q^{m-j})}{4}-1}\}$.

(xviii) $C_{2abp^iq^j} = \{2abp^iq^j, 2abp^iq^j l, \ldots, 2abp^iq^j l^{\frac{\phi(p^{n-i}q^{m-j})}{4}-1}\}$.

where a and b are defined in Lemma2.2.

Proof.Proof follows on the same lines as in Theorem2.4.[5].

Theorem2.5. The 8mn+4n+4m+2 primitive idempotents corresponding to cyclotomic cosets $C_0, C_{p^nq^m}, C_{p^nq^k}, C_{2p^nq^k}, C_{ap^nq^k}, C_{2ap^nq^k}$ $C_{p^jq^m}, C_{bp^jq^m}, C_{2p^jq^m}$ and $C_{2bp^jq^m}$, $0 \le j \le n-1, 0 \le k \le m-1$ in $R_{2p^nq^m}$ are

(i) $\theta_0(x) = \dfrac{1}{2p^nq^m}(1 + x + x^2 + \ldots + x^{2p^nq^m-1})$.

(ii) $$\theta_{p^nq^m}(x) = \frac{1}{2p^nq^m}\{1 - \sigma_{(n,m)}(x) + \sum_{k=0}^{m-1}\sigma_{2(n,k)}(x) + \sigma_{2a(n,k)}(x) - \sigma_{(n,k)}(x) - \sigma_{a(n,k)}(x)$$

$$+ \sum_{(i,r)=(0,0)}^{(n-1,m-1)}(\sigma_{2(i,r)}(x) + \sigma_{2b(i,r)}(x) + \sigma_{2a(i,r)}(x) + \sigma_{2ab(i,r)}(x) - \sigma_{(i,r)}(x) - \sigma_{b(i,r)}(x) - \sigma_{(i,r)}(x) - \sigma_{b(i,r)}(x)$$

$$+ \sum_{j=0}^{n-1}\sigma_{2(j,m)}(x) + \sigma_{2b(j,m)}(x) - \sigma_{(j,m)}(x) - \sigma_{b(j,m)}(x).$$

(iii) For $0 \le k \le m-1$,

$$\theta_{p^n q^k}(x) = -\frac{\phi(q^{m-k})}{4p^n q^m}\{\sigma_{(n,m)}(x) - 1 + \sum_{(i,r)=(0,m-k)}^{(n-1,m-1)}(\sigma_{(i,r)}(x) + \sigma_{b(i,r)}(x) + \sigma_{a(i,r)}(x) + \sigma_{ab(i,r)}(x))\}$$

$$+\frac{1}{4p^n q^m}\{\phi(q^{m-k})[\sum_{(i,r)=(0,m-k)}^{(n-1,m-1)}(\sigma_{2(i,r)}(x) + \sigma_{2b(i,r)}(x) + \sigma_{2a(i,r)}(x) + \sigma_{2ab(i,r)}(x))$$

$$-\sum_{(i,r)=(n,m-k)}^{(n,m-1)}\sigma_{(n,r)}(x) + \sigma_{a(n,r)}(x) - \sum_{(i,r)=(0,m)}^{(n-1,m)}(\sigma_{(i,r)}(x) + \sigma_{b(i,r)}(x))$$

$$+\sum_{(i,r)=(n,m-k)}^{(n,m-1)}\sigma_{2(n,r)}(x) + \sigma_{2a(n,r)}(x) + \sum_{(i,r)=(0,m)}^{(n-1,m)}(\sigma_{2(i,r)}(x) + \sigma_{2b(i,r)}(x))]$$

$$+ 2q^{m-k-1}[\sum_{i=0}^{n-1}\eta_1^*(\sigma_{2(i,m-k-1)}(x) + \sigma_{2b(i,m-k-1)}(x)) + \eta_0^*(\sigma_{2a(i,m-k-1)}(x) + \sigma_{2ab(i,m-k-1)}(x))$$

$$+\eta_1^*(\sigma_{2(n,m-k-1)}(x)) + \eta_0^*(\sigma_{2a(n,m-k-1)}(x))]$$

$$+ 2q^{m-k-1}[\sum_{i=0}^{n-1}\xi_1^*(\sigma_{(i,m-k-1)}(x) + \sigma_{b(i,m-k-1)}(x)) + \xi_0^*(\sigma_{a(i,m-k-1)}(x) + \sigma_{ab(i,m-k-1)}(x))$$

$$+\xi_1^*(\sigma_{(n,m-k-1)}(x)) + \xi_0^*(\sigma_{a(n,m-k-1)}(x))\}$$

Similarly we can find $\theta_{ap^n q^k}(x), \theta_{2ap^n q^k}(x)$.

(iv) For $0 \le j \le n-1$

$$\theta_{p^j q^m}(x) = \frac{1}{2p^{j+1}q^m}\{\sum_{r=0}^{m-1}\{\eta_1(\sigma_{2(n-j-1,r)}(x) + \sigma_{2a(n-j-1,r)}(x)) + \eta_0(\sigma_{2b(n-j-1,r)}(x) + \sigma_{2ab(n-j-1,r)}(x))$$

$$+\sum_{r=0}^{m-1}\{\xi_1(\sigma_{(n-j-1,r)}(x) + \sigma_{a(n-j-1,r)}(x)) + \xi_0(\sigma_{b(n-j-1,r)}(x) + \sigma_{ab(n-j-1,r)}(x))\}$$

$$+\frac{\phi(p^{n-j})}{4p^n q^m}\sum_{r=0}^{m-1}\sigma_{2(n,r)}(x) + \sigma_{2a(n,r)}(x) - \sigma_{(n,r)}(x) - \sigma_{a(n,r)}(x) + \sigma_{2(n,m)}(x) - \sigma_{(n,m)}(x)$$

$$+\frac{\phi(p^{n-j})}{4p^n q^m}\sum_{(i,r)=(n-j,0)}^{(n-1,m-1)}(\sigma_{2(i,r)}(x) + \sigma_{2b(i,r)}(x) + \sigma_{2a(i,r)}(x) + \sigma_{2ab(i,r)}(x)$$

$$-\frac{\phi(p^{n-j})}{4p^n q^m}\sum_{(i,r)=(n-j,0)}^{(n-1,m-1)}(\sigma_{(i,r)}(x) + \sigma_{b(i,r)}(x) + \sigma_{a(i,r)}(x) + \sigma_{ab(i,r)}(x)$$

$$+\frac{1}{2p^{j+1}q^m}(\eta_1\sigma_{2(i,m)}(x) + \eta_0\sigma_{2b(i,m)}(x) + \xi_1\sigma_{(i,m)}(x) + \xi_0\sigma_{b(i,m)}(x))$$

$$+\frac{\phi(p^{n-j})}{4p^n q^m}\sum_{i=n-j}^{n-1}(\sigma_{2(i,m)}(x) + \sigma_{2b(i,m)}(x) - \sigma_{(i,m)}(x) - \sigma_{b(i,m)}(x))$$

Similarly we can find $\theta_{bp^j q^m}(x), \theta_{2p^j q^m}(x), \theta_{2bp^j q^m}(x)$.

(v) For $0 \le j \le n-1,\ 0 \le k \le m-1$

$$\theta_{p^j q^k}(x) = \frac{1}{2p^n q^m}\{\frac{1}{p^j q^k}\sum_{(i,r)=(0,0)}^{(n-j-1,m-k-1)}[A^*_{(i+j,r+k)}\sigma_{a(i,r)}(x) + B^*_{(i+j,r+k)}\sigma_{ab(i,r)}(x)] +$$

$$\frac{1}{p^j q^k}\sum_{(i,r)=(0,0)}^{(n-j-1,m-k-1)}[A_{(i+j,r+k)}\sigma_{2a(i,r)}(x) + B_{(i+j,r+k)}\sigma_{2ab(i,r)}(x)]$$

$$+\frac{1}{p^j q^k}\sum_{(i,r)=(0,0)}^{(n-j-1,m-k-1)}[X^*_{i+j,r+k)}\sigma_{b(i,r)}(x)+Y^*_{i+j,r+k)}\sigma_{(i,r)}(x)]$$

$$+\frac{1}{p^j q^k}\sum_{(i,r)=(0,0)}^{(n-j-1,m-k-1)}[X_{(i+j,r+k)}\sigma_{2b(i,r)}(x)+Y_{(i+j,r+k)}\sigma_{2(i,r)}(x)]$$

$$+\frac{\phi(p^{n-j})q^{m-k-1}}{2}\sum_{(i,r)=(n-j,m-k-1)}^{(n-1,m-k-1)}\{\xi_0^*(\sigma_{a(i,m-k-1)}(x)+\sigma_{ab(i,m-k-1)}(x))+\xi_1^*(\sigma_{(i,m-k-1)}(x)+\sigma_{b(i,m-k-1)}(x))\}$$

$$+\frac{\phi(p^{n-j})q^{m-k-1}}{2}[\sum_{(i,r)=(n-j,m-k-1)}^{(n-1,m-k-1)}\{\eta_0^*(\sigma_{2a(i,m-k-1)}(x)+\sigma_{2ab(i,m-k-1)}(x))+\eta_1^*(\sigma_{2(i,m-k-1)}(x)+\sigma_{2b(i,m-k-1)}(x))\}+\eta_1^*$$

$$\sigma_{2(n,m-k-1)}(x)+\eta_0^*\,\sigma_{2a(n,m-k-1)}(x)+\xi_1^*\,\sigma_{(n,m-k-1)}(x)+\xi_0^*\,\sigma_{a(n,m-k-1)}(x)]$$

$$+p^{n-j-1}\frac{\phi(q^{m-k})}{2}\sum_{(i,r)=(n-j-1,m-k)}^{(n-j-1,m)}[\xi_1(\sigma_{(n-j-1,r)}(x)+\sigma_{a(n-j-1,r)}(x))+\xi_o(\sigma_{b(n-j-1,r)}(x)+\sigma_{ab(n-j-1,r)}(x))]$$

$$+p^{n-j-1}\frac{\phi(q^{m-k})}{2}\sum_{(i,r)=(n-j-1,m-k)}^{(n-j-1,m)}[\eta_1(\sigma_{2(n-j-1,r)}(x)+\sigma_{2a(n-j-1,r)}(x))+\eta_o(\sigma_{2b(n-j-1,r)}(x)+\sigma_{2ab(n-j-1,r)}(x))]$$

$$-\frac{\phi(p^{n-j}q^{m-k})}{4}[\sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)}(\sigma_{(i,,r)}(x)+\sigma_{b(i,r)}(x)+\sigma_{a(i,,r)}(x)+\sigma_{ab(i,r)}(x))]$$

$$+\frac{\phi(p^{n-j}q^{m-k})}{4}[\sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)}(\sigma_{2(i,,r)}(x)+\sigma_{2b(i,r)}(x)+\sigma_{2a(i,,r)}(x)+\sigma_{2ab(i,r)}(x))$$

$$+\sum_{(i,r)=(n,m-k)}^{(n,m-1)}\sigma_{2(n,,r)}(x)+\sigma_{2a(n,,r)}(x)-\sum_{(i,r)=(n,m-k)}^{(n,m-1)}\sigma_{(n,,r)}(x)+\sigma_{a(n,,r)}(x)$$

$$-\sum_{(i,r)=(n-j,m)}^{(n-1,m)}(\sigma_{(i,m)}(x)+\sigma_{b(i,m)}(x)+\sum_{(i,r)=(n-j,m)}^{(n-1,m)}(\sigma_{2(i,m)}(x)+\sigma_{2b(i,m)}(x))+2]\}$$

. Similarly we can find $\theta_{bp^jq^k}(x)$, $\theta_{ap^jq^k}(x)$, $\theta_{abp^jq^k}(x)$, $\theta_{2p^jq^k}(x)$, $\theta_{2ap^jq^k}(x)$, $\theta_{2bp^jq^k}(x)$ and $\theta_{2abp^jq^k}(x)$.

where $A^*_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m-1}(\frac{-1+r+\gamma+\delta}{4})$, $B^*_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m-1}(\frac{-1+r-\delta-\gamma}{4})$

$X^*_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m-1}(\frac{-1-r-\gamma+\delta}{4})$, $Y^*_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m-1}(\frac{-1-r+\gamma-\delta}{4})$

$A_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m-1}(\frac{1+r+\gamma+\delta}{4})$, $B_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m-1}(\frac{1+r-\delta-\gamma}{4})$

$X_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m-1}(\frac{1-r-\gamma+\delta}{4})$, $Y_{(n-1,m-1)}=\mathbf{p}^{n-1}q^{m1}(\frac{1-r+\gamma-\delta}{4})$

, $\eta_0=\frac{-1+\sqrt{-p}}{2}$, $\eta_1=\frac{-1-\sqrt{-p}}{2}$, $\eta_o^*=\frac{-1+\sqrt{-q}}{2}$, $\eta_1^*=\frac{-1-\sqrt{-q}}{2}$,

$\xi_0=\frac{1+\sqrt{-p}}{2}$, $\xi_1=\frac{1-\sqrt{-p}}{2}$, $\xi_0^*=\frac{1+\sqrt{-q}}{2}$ and $\xi_1^*=\frac{1-\sqrt{-q}}{2}$.

where $r^2=-q$, $\gamma^2=-p$, $\delta^2=pq$.

5. References

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