A unified study of Fourier series involving the Aleph-function and the Kampé de Fériet's function

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Abstract : Recently Yashwant Singh et al [7] have studied Fourier series involving the I-function defined by V.P. Saxena [6]. Motivated by this work, we make an application of an integral involving sine function, exponential function, the product of Aleph-function of one variable and Kampé de Fériet's function. We also evaluate a multiple integral involving the Aleph-function to make its application to derive a multiple exponential Fourier series. Several particular cases are also given at the end.

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1. Introduction and notations

The Aleph- function , introduced by Südland [8] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(z\ \middle|\ \begin{array}{c}(a_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ \\ (b_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right)$$

$$=\frac{1}{2\pi\omega}\int_L \Omega^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}(s)z^{-s}\mathrm{d}s \qquad (1.1)$$

for all z different to 0 and

$$\Omega^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_js)\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}c_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)} \qquad (1.2)$$

For convergence conditions and other details of Aleph-function , see Südland et al [8]. The Kampé de Fériet hypergeometric function will represented as follows.

$$K^{E;F;F'}_{G;H;H'}\left(\begin{array}{c}(e);(f);(f');x\\ \\ (g);(h);(h');y\end{array}\right)=\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}\frac{\prod_{k=1}^{E}(e_k)_{r+t}\prod_{k=1}^{F}(f_k)_r\prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t}\prod_{k=1}^{H}(h_k)_r\prod_{k=1}^{H'}(h'_k)_t}\times\frac{x^ry^t}{r!t!} \qquad (1.3)$$

For further detail see Appell and Kampé de Fériet [1]. For brevity , we shall use the following notations.

$$\epsilon=\frac{\prod_{k=1}^{E}(e_k)_{r+t}\prod_{k=1}^{F}(f_k)_r\prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t}\prod_{k=1}^{H}(h_k)_r\prod_{k=1}^{H'}(h'_k)_t}$$

$$\epsilon_1 = \frac{\prod_{k_1=1}^{E_1}(e_{1k_1})_{r_1+t_1}\prod_{k_1=1}^{F_1}(f_{1k_1})_{r_1}\prod_{k_1=1}^{F'_1}(f'_{1k_1})_{t_1}}{\prod_{k_1=1}^{G_1}(g_{1k_1})_{r_1+t_1}\prod_{k_1=1}^{H_1}(h_{1k_1})_{r_1}\prod_{k_1=1}^{H'_1}(h'_{1k_1})_{t_1}}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\epsilon_n = \frac{\prod_{k_n=1}^{E_n}(e_{nk_n})_{r_n+t_n}\prod_{k_n=1}^{F_n}(f_{nk_n})_{r_n}\prod_{k_n=1}^{F'_n}(f'_{nk_n})_{t_n}}{\prod_{k_n=1}^{G_n}(g_{nk_n})_{r_n+t_n}\prod_{k_n=1}^{H_n}(h_{nk_n})_{r_n}\prod_{k_n=1}^{H'_n}(h'_{nk_n})_{t_n}}$$

Mishra [3] has evaluated the following integral :

$$\int_0^\pi (sinx)^{w-1}e^{imx}\, {}_pF_q\left(\begin{matrix}(\alpha_p);\\(\beta_q);\end{matrix} C(sinx)^{2h}\right)\mathrm{d}x = \frac{\pi e^{im\pi/2}}{2^{w-1}}\sum_{r=0}^{\infty}\frac{(\alpha_p)_r C^r\Gamma(w+2hr)}{(\beta_q)_r 4^{hr}\Gamma(\frac{w+2hr\pm m+1}{2})r!} \tag{1.4}$$

where $(\alpha)_p$ denotes $\alpha_1,\cdots,\alpha_p$; $\Gamma(a\pm b)$ represents $\Gamma(a+b),\Gamma(a-b)$; h is a positive integer ; $p<q$ and $Re(w)>0$.

We have the following results :

$$\int_0^\pi e^{i(m-n)x}\,\mathrm{d}x = \pi\delta_{mn}\,;\quad \int_0^\pi e^{imx}sinnx\,\mathrm{d}x = i\frac{\pi}{2}\delta_{mn}\qquad \text{where } \delta_{mn}=1 \text{ if } m=n \text{ ,0 else.} \tag{1.7}$$

$$\int_0^\pi e^{imx}cosnx\,\mathrm{d}x = \pi\upsilon_{mn}\quad \text{where } \upsilon_{mn} = \frac{1}{2} \text{ if } m=n\neq 0\,,1 \text{ if } m=n=0\,,0 \text{ else.} \tag{1.8}$$

2. Main results

The integrals to be evaluate are :

$$\text{a)}\int_0^\pi (sinx)^{w-1}e^{imx}\,K_{G;H;H'}^{E;F;F'}\left(\begin{matrix}(\text{ e });(\text{ f });(\text{ f' });\alpha(sinx)^{2\rho}\\ \\(\text{ g });(\text{ h });(\text{ h' });\beta(sinx)^{2\gamma}\end{matrix}\right)\times$$

$$\aleph_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}\left(z(sinx)^{2\sigma}\;\middle|\;\begin{matrix}(\text{ a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ \\(\text{ b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right)\mathrm{dx} = \frac{\pi^{\frac{1}{2}}e^{im\pi/2}}{2^{w-1}}\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}\epsilon\times$$

$$\frac{\alpha^r\beta^t}{4^{(\rho r+\gamma t)}r!t!}\aleph_{p_i+1,q_i+2,c_i;r}^{m,\mathfrak{n}+1}\left(\frac{z}{4^\sigma}\;\middle|\;\begin{matrix}(\text{ a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r},(1-w-2\rho r-2\gamma t,2\sigma)\\ \\(\text{ b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r},(\frac{1-w-2\rho r-2\gamma t\pm m}{2},\sigma)\end{matrix}\right) \tag{2.1}$$

provided that $Re(w)>0\,,\alpha,\beta,\rho,\gamma,\sigma,z$ are positive integers. The numbers c_i are positives.

$$|argz| < \frac{1}{2}\pi\Omega \quad \text{Where } \Omega = \sum_{j=1}^{m}\beta_j + \sum_{j=1}^{\mathfrak{n}}\alpha_j - c_i(\sum_{j=m+1}^{q_i}\beta_{ji} + \sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}) > 0 \quad \text{with } i = 1,\cdots,r$$

b) $$\int_0^{\pi}\cdots\int_0^{\pi}\prod_{j=1}^{n}(sinx)^{w_j-1}e^{im_jx_j}\,K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j}\left(\begin{array}{c}(\mathrm{e}_j),(f_j),(f'_j);\alpha_j(sinx_j)^{2\rho_j}\\ (\mathrm{g}_j),(h_j),(h'_j);\beta_j(sinx_j)^{2\gamma_j}\end{array}\right)$$

$$\times\aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(\mathrm{z}\prod_{j=1}^{n}(sinx_j)^{2\sigma_j}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ (\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)\mathrm{d}x_1\cdots\mathrm{d}x_r =$$

$$= \prod_{j=1}^{n}\frac{\pi e^{im_j\pi/2}}{2^{w_j-1}}\sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty}\prod_{j=1}^{n}\epsilon_j\frac{\alpha_j^{r_j}\beta_j^{t_j}}{4^{(\rho_jr_j+\gamma_jt_j)}r_j!t_j!}\times$$

$$\times\aleph^{m,\mathfrak{n}+n}_{p_i+n,q_i+2n,c_i;r}\left(\frac{z}{4^{\sigma_1+\cdots+\sigma_n}}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r},\\ (\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r},\end{array}\right.\right.$$

$$\left.\begin{array}{c}(1-\mathrm{w}_1-2\rho_1r_1-2\gamma_1t_1,2\sigma_1),\cdots,(1-w_n-2\rho_nr_n-2\gamma_nt_n,2\sigma_n)\\ \cdots\\ (\frac{1-w_1-2\rho_1r_1-2\gamma_1t_1\pm m_1}{2},\sigma_1),\cdots,(\frac{1-w_n-2\rho_nr_n-2\gamma_nt_n\pm m_n}{2},\sigma_n)\end{array}\right) \tag{2.2}$$

provided that all the conditions of (2.1) are satisfied and $Re(w_j) > 0$, $\alpha_j,\beta_j,\rho_j,\gamma_j,\sigma_j,z_j$ are positives integers for $j = 1,\cdots,n$. The numbers c_i are positives.

Proof

To prove (2.1) , express the Aleph-function in the Mellin-Barnes integral with the help of (1.1) and the Kampé de Fériet function in double serie with the help of (1.3). We change the order of integration and summation, wich is permissible under the conditions stated, now evaluate the x-integral with the help of (1.4) and reinterpreting the Mellin-Barnes contour integral in the form of Aleph-function, we get the desire result (2.1).

The integral (2.2) is obtained by the similar method.

3. Exponential Fourier series

Let $$f^{(1)}(x) = (sinx)^{w-1}K^{E;F;F'}_{G;H;H'}\left(\begin{array}{c}(\mathrm{e});(\mathrm{f});(\mathrm{f'});\alpha(sinx)^{2\rho}\\ (\mathrm{g});(\mathrm{h});(\mathrm{h'});\beta(sinx)^{2\gamma}\end{array}\right)$$

$$\times\aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(z(sinx)^{2\sigma}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ (\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right) = \sum_{p=-\infty}^{\infty}A_pe^{-ipx} \tag{3.1}$$

where $f^{(1)}(x)$ is a continuous function and bounded variation with interval $(0,\pi)$. Now , multiplied by e^{imx} both sides in (3.1) and integrating it with respect x from 0 to π. Use the first relation of (1.7) and (2.1) , we get :

$$A_p = \frac{e^{ip\pi/2}}{2^{w-1}} \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \epsilon \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!t!} \times$$

$$\aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+2,c_i;r}\left(\frac{z}{4^\sigma}\left|\begin{matrix}(\text{ a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, (1-w-2\rho r-2\gamma t, 2\sigma)\\ (\text{ b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, (\frac{1-w-2\rho r-2\gamma t\pm m}{2}, \sigma)\end{matrix}\right.\right) \tag{3.2}$$

Use (3.1) and (3.2) , we obtain the following exponential Fourier serie

$$(2sinx)^{w-1}\, K^{E;F;F'}_{G;H;H'}\left(\begin{matrix}(\text{ e }) ; (\text{ f }) ; (\text{ f' }) ; \alpha(sinx)^{2\rho}\\ (\text{ g }) ; (\text{ h }) ; (\text{ h' }) ; \beta(sinx)^{2\gamma}\end{matrix}\right)$$

$$\times\, \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(z(sinx)^{2\sigma}\left|\begin{matrix}(\text{ a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ (\text{ b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right) =$$

$$= \sum_{p=-\infty}^{\infty}\sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \epsilon e^{ip(\pi/2-x)} \times \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!t!}$$

$$\times\, \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+2,c_i;r}\left(\frac{z}{4^\sigma}\left|\begin{matrix}(\text{ a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, (1-w-2\rho r-2\gamma t, 2\sigma)\\ (\text{ b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, (\frac{1-w-2\rho r-2\gamma t\pm m}{2}, \sigma)\end{matrix}\right.\right) \tag{3.3}$$

4. Cosine Fourier series

Let $f^{(2)}(x) = (sinx)^{w-1}\, K^{E;F;F'}_{G;H;H'}\left(\begin{matrix}(\text{ e }) ; (\text{ f }) ; (\text{ f' }) ; \alpha(sinx)^{2\rho}\\ (\text{ g }) ; (\text{ h }) ; (\text{ h' }) ; \beta(sinx)^{2\gamma}\end{matrix}\right)$

$$\times\, \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(z(sinx)^{2\sigma}\left|\begin{matrix}(\text{ a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ (\text{ b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right) = \frac{B_0}{2} + \sum_{p=1}^{\infty} B_p cospx \tag{4.1}$$

Integrating it with respect x from 0 to π , we have :

$$\frac{B_0}{2} = \frac{1}{\pi^{\frac{1}{2}}} \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \epsilon \frac{\alpha^r \beta^t}{r!t!}$$

$$\times \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+1,c_i;r}\left(z\left|\begin{matrix}(\text{ a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, (\frac{2-w}{2}-\rho r-\gamma t, 2\sigma)\\ (\text{ b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, (\frac{1-w}{2}-2\rho r-2\gamma t, \sigma)\end{matrix}\right.\right) \tag{4.2}$$

Multiplying the both sides in (4.1) by e^{imx} and integrating it with respect x from 0 to π and use the equations (1.7) , (1.8) and (2.1), we obtain.

$$B_p = \frac{e^{ip\pi/2}}{2^{w-1}} \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \epsilon \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!t!} \times$$

$$\times \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+2,c_i;r}\left(\frac{z}{4^\sigma}\,\middle|\, \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, (1-w-2\rho r-2\gamma t, 2\sigma) \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, (\frac{1-w-2\rho r-2\gamma t\pm m}{2}, \sigma) \end{matrix}\right) \tag{4.3}$$

Use the equations (4.2) , (4.3) and (4.1), we obtain the following cosine Fourier serie

$$(sinx)^{w-1} K^{E;F;F'}_{G;H;H'}\left(\begin{matrix} (\mathrm{e}) ; (\mathrm{f}) ; (\mathrm{f'}) ; \alpha(sinx)^{2\rho} \\ (\mathrm{g}) ; (\mathrm{h}) ; (\mathrm{h'}) ; \beta(sinx)^{2\gamma} \end{matrix}\right)$$

$$\times \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(z(sinx)^{2\sigma}\,\middle|\, \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix}\right) = \frac{1}{\pi^{\frac{1}{2}}} \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \epsilon \frac{\alpha^r \beta^t}{r!t!}$$

$$\times \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+1,c_i;r}\left(z\,\middle|\, \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, (\frac{2-w}{2}-\rho r-\gamma t, 2\sigma) \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, (\frac{1-w}{2}-2\rho r-2\gamma t, \sigma) \end{matrix}\right)$$

$$+ \sum_{p=-\infty}^{\infty} \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \frac{\epsilon}{2^{w-2}} e^{ip\pi/2} cospx \times \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!t!}$$

$$\times \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+2,c_i;r}\left(\frac{z}{4^\sigma}\,\middle|\, \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, (1-w-2\rho r-2\gamma t, 2\sigma) \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, (\frac{1-w-2\rho r-2\gamma t\pm m}{2}, \sigma) \end{matrix}\right) \tag{4.4}$$

5. Sine Fourier series

Let $g^{(3)}(x) = (sinx)^{w-1} K^{E;F;F'}_{G;H;H'}\left(\begin{matrix} (\mathrm{e}) ; (\mathrm{f}) ; (\mathrm{f'}) ; \alpha(sinx)^{2\rho} \\ (\mathrm{g}) ; (\mathrm{h}) ; (\mathrm{h'}) ; \beta(sinx)^{2\gamma} \end{matrix}\right)$

$$\times \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(z(sinx)^{2\sigma}\,\middle|\, \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix}\right) = \sum_{p=-\infty}^{\infty} C_p sinpx \tag{5.1}$$

Multiplying the both sides in (5.1) by e^{imx} and integrating it with respect x from 0 to π and use the equations (1.8) , and (2.1), we obtain.

$$C_p = \frac{e^{ip\pi/2}}{2^{w-1}} \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \epsilon \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!t!} \times$$

$$\times \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+2,c_i;r}\left(\frac{z}{4^\sigma}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r},(1-w-2\rho r-2\gamma t,2\sigma)\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r},(\frac{1-w-2\rho r-2\gamma t\pm m}{2},\sigma)\end{array}\right.\right) \tag{5.2}$$

Use the equations (5.1) and (5.2) , we get the following sine Fourier serie :

$$(2sinx)^{w-1}\,K^{E;F;F'}_{G;H;H'}\left(\begin{array}{c}(\mathrm{e});(\mathrm{f});(\mathrm{f'});\alpha(sinx)^{2\rho}\\(\mathrm{g});(\mathrm{h});(\mathrm{h'});\beta(sinx)^{2\gamma}\end{array}\right)$$

$$\times \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(z(sinx)^{2\sigma}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)$$

$$= -2i\sum_{p=-\infty}^{\infty}\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}\epsilon e^{ip\pi/2}sinpx\,\frac{\alpha^r\beta^t}{4^{(\rho r+\gamma t)}r!t!}$$

$$\times \aleph^{m,\mathfrak{n}+1}_{p_i+1,q_i+2,c_i;r}\left(\frac{z}{4^\sigma}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r},(1-w-2\rho r-2\gamma t,2\sigma)\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r},(\frac{1-w-2\rho r-2\gamma t\pm m}{2},\sigma)\end{array}\right.\right) \tag{5.3}$$

6. Multiple exponential Fourier series

Consider $f(x_1,\cdots,x_n)$ a function continuous and bounded variations in the domain $(0,\pi)\times\cdots\times(0,\pi)$ and

$$f(x_1,\cdots,x_n)=\prod_{j=1}^{n}(sinx)^{w_j-1}\,K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j}\left(\begin{array}{c}(\mathrm{e}_j),(f_j),(f'_j);\alpha_j(sinx_j)^{2\rho_j}\\(\mathrm{g}_j),(h_j),(h'_j);\beta_j(sinx_j)^{2\gamma_j}\end{array}\right)$$

$$\times\aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(\mathrm{z}\prod_{j=1}^{n}(sinx_j)^{2\sigma_j}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)=$$

$$=\sum_{p_1,\cdots,p_n=-\infty}^{\infty}A_{p_1,\cdots,p_n}\,e^{-i(p_1x_1+\cdots p_nx_n)} \tag{6.1}$$

We fix $x_1,\cdots,x_{n-1}$ and multipling the both sides in (6.1) by $e^{im_nx_n}$ and integrating with respect to x_n from 0 to π , we obtain :

$$\prod_{j=1}^{n-1}(sinx)^{w_j-1}K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j}\left(\begin{array}{c}(\mathrm{e}_j),(f_j),(f'_j);\alpha_j(sinx_j)^{2\rho_j}\\(\mathrm{g}_j),(h_j),(h'_j);\beta_j(sinx_j)^{2\gamma_j}\end{array}\right)$$

$$\times\int_0^\pi(sinx_n)^{w_n-1}e^{im_nx_n}K^{E_n;F_n;F'_n}_{G_n;H_n;H'_n}\left(\begin{array}{c}(\mathrm{e}_n),(f_n),(f'_n);\alpha_n(sinx_n)^{2\rho_n}\\(\mathrm{g}_n),(h_n),(h'_n);\beta_n(sinx_n)^{2\gamma_n}\end{array}\right)$$

$$\times \aleph_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}\left(\mathrm{z}\prod_{j=1}^{n}(sinx_j)^{2\sigma_j} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right) \mathrm{d}x_n$$

$$= \sum_{p_1,\cdots,p_{n-1}=-\infty}^{\infty} A_{p_1\cdots,p_{n-1}}\, e^{-i(p_1x_1+\cdots+p_nx_n)} + \sum_{p_n=-\infty}^{\infty}\int_0^{\pi} e^{i(m_n-p_n)}\,\mathrm{d}x_n$$

Use the first relation of (1.7) and (2.1) , from (6.2) , we get :

$$A_{p_1,\cdots,p_n} = \sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty}\prod_{j=1}^{n}\epsilon_j\,\frac{e^{ip_j\pi/2}}{2^{w_j-1}}\times\frac{\alpha_j^{r_j}\beta_j^{t_j}}{4^{(\rho_jr_j+\gamma_jt_j)}r_j!t_j!}$$

$$\times \aleph_{p_i+n,q_i+2n,c_i;r}^{m,\mathfrak{n}+n}\left(\frac{z}{4^{\sigma_1+\cdots+\sigma_n}} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, \end{matrix}\right.$$

$$\left.\begin{matrix}(1 - \mathrm{w}_1 - 2\rho_1r_1 - 2\gamma_1t_1, 2\sigma_1), \cdots, (1 - w_n - 2\rho_nr_n - 2\gamma_nt_n, 2\sigma_n) \\ (\frac{1-w_1-2\rho_1r_1-2\gamma_1t_1\pm m_1}{2},\sigma_1), \cdots, (\frac{1-w_n-2\rho_nr_n-2\gamma_nt_n\pm m_n}{2}, \sigma_n)\end{matrix}\right) \tag{6.3}$$

Using (6.1) and (6.3) , we obtain the multiple exponential Fourier serie.

$$\prod_{j=1}^{n}(sinx)^{w_j-1}\, K_{G_j;H_j;H_j'}^{E_j;F_j;F_j'}\left(\begin{matrix} (\mathrm{e}_j), (f_j), (f_j'); \alpha_j(sinx_j)^{2\rho_j} \\ (\mathrm{g}_j), (h_j), (h_j'); \beta_j(sinx_j)^{2\gamma_j}\end{matrix}\right)$$

$$\times \aleph_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}\left(\mathrm{z}\prod_{j=1}^{n}(sinx_j)^{2\sigma_j} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right)$$

$$= \sum_{p_1,\cdots,p_n=-\infty}^{\infty}\ \sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty}\prod_{j=1}^{n}\epsilon_j\, e^{-ip_jx_j}\times\frac{e^{ip_j\pi/2}}{2^{w_j-1}}\times\frac{\alpha_j^{r_j}\beta_j^{t_j}}{4^{(\rho_jr_j+\gamma_jt_j)}r_j!t_j!}$$

$$\times \aleph_{p_i+n,q_i+2n,c_i;r}^{m,\mathfrak{n}+n}\left(\frac{z}{4^{\sigma_1+\cdots+\sigma_n}} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix}\right.$$

$$\left.\begin{matrix},(1 - \mathrm{w}_1 - 2\rho_1r_1 - 2\gamma_1t_1, 2\sigma_1), \cdots, (1 - w_n - 2\rho_nr_n - 2\gamma_nt_n, 2\sigma_n) \\ ,(\frac{1-w_1-2\rho_1r_1-2\gamma_1t_1\pm m_1}{2},\sigma_1), \cdots, (\frac{1-w_n-2\rho_nr_n-2\gamma_nt_n\pm m_n}{2}, \sigma_n)\end{matrix}\right) \tag{6.4}$$

7. Particular cases

The Aleph-function is a generalization of I-function and H-function , for more details , see D.Kumar et al [4 , 5]. We obtain similar results with I-function and H-function of one variable , see Y.A. Singh et al [7].

Setting $\beta_1, \cdots, \beta_n = 0$ in (2.2) , we get the following integral :

$$\int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{n} (sinx)^{w_j-1} e^{im_j x_j} \; {}_{E_j+F_j}K_{G_j+H_j}\left(\alpha_j (sinx_j)^{2\rho_j} \middle| \begin{matrix} (\mathrm{e}_j), (f_j) \\ (\mathrm{g}_j), (h_j) \end{matrix} \right)$$

$$\times \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(\mathrm{z} \prod_{j=1}^{n} (sinx_j)^{2\sigma_j} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_r$$

$$= \prod_{j=1}^{n} \frac{\pi e^{im_j\pi/2}}{2^{w_j-1}} \sum_{r_1,\cdots,r_n=0}^{\infty} \prod_{j=1}^{n} \mathfrak{E}_j \frac{\alpha_j^{r_j}}{4^{\rho_j r_j} r_j!}$$

$$\times \aleph^{m,\mathfrak{n}+n}_{p_i+n,q_i+2n,c_i;r}\left(\frac{z}{4^{\sigma_1+\cdots+\sigma_n}} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right.$$

$$\left. \begin{matrix} ,(1 - \mathrm{w}_1 - 2\rho_1 r_1 - 2\gamma_1 t_1, 2\sigma_1), \cdots, (1 - w_n - 2\rho_n r_n - 2\gamma_n t_n, 2\sigma_n) \\ ,(\frac{1-w_1-2\rho_1 r_1-2\gamma_1 t_1 \pm m_1}{2}, \sigma_1), \cdots, (\frac{1-w_n-2\rho_n r_n-2\gamma_n t_n \pm m_n}{2}, \sigma_n) \end{matrix} \right) \quad (7.1)$$

with $\mathfrak{E}_j = \frac{\prod_{k_j=1}^{E_j}(e_{jk_j})_{r_j} \prod_{k_j=1}^{F_j}(f_{jk_j})_{r_j}}{\prod_{k_j=1}^{G_j}(g_{jk_j})_{r_j} \prod_{k_j=1}^{H_j}(h_{jk_j})_{r_j}} \quad , j = 1, \cdots, n$

If $\alpha_1 = \cdots = \alpha_n = 0$ in (7.1), we get :

$$\int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{n} (sinx)^{w_j-1} e^{im_j x_j}$$

$$\times \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(\mathrm{z} \prod_{j=1}^{n} (sinx_j)^{2\sigma_j} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_r$$

$$= \prod_{j=1}^{n} \frac{\pi e^{im_j\pi/2}}{2^{w_j-1}} \aleph^{m,\mathfrak{n}+1}_{p_i+n,q_i+n,c_i;r}\left(\frac{z}{4^{\sigma_1+\cdots+\sigma_n}} \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r}, \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r}, \end{matrix} \right.$$

$$\left. \begin{matrix} (1 - \mathrm{w}_1, 2\rho_1), \cdots, (1 - w_n, 2\rho_n) \\ (\frac{1-w_1 \pm m_1}{2}, \sigma_1), \cdots, \frac{1-w_n \pm m_n}{2}, \sigma_n) \end{matrix} \right) \quad (7\ 2)$$

Remark : We obtain the similar formulas with multivariable h-function , see R. C. Chandel [2]

8. Conclusion

The aleph-function, presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as I-function ,Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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