Certain finite double integrals involving the hypergeometric function and Aleph-function

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Abstract : The aim of this document is to evaluate four finite double integrals involving the product of two hypergeometric functions and the Aleph-function. At the end of this paper , we evaluate few particular cases.

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1 Introduction and notations

The Aleph- function , introduced by Südland [8] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}\left(\mathrm{z}\ \middle|\ \begin{matrix}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L\Omega_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}(s)z^{-s}\mathrm{d}s \tag{1.1}$$

for all z different to 0 and

$$\Omega_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}(s)=\frac{\prod_{j=1}^m\Gamma(b_j+\beta_js)\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j-A_js)}{\sum_{i=1}^r c_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.2}$$

With

$$|argz|<\frac{1}{2}\pi\Omega \quad \text{Where } \Omega=\sum_{j=1}^m\beta_j+\sum_{j=1}^{\mathfrak{n}}\alpha_j-c_i\Big(\sum_{j=m+1}^{q_i}\beta_{ji}+\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}\Big)>0 \quad \text{with } i=1,\cdots,r$$

For convergence conditions and other details of Aleph-function , see Südland et al [8].

We shall use notation the following :

$A=(a_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}$ and $B=(b_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}$

For more details ,see D.Kumar et all [3].

2 Hypergeometric function

We have the following results , see Rathie et al [7]

$$\int_0^1 x^{\rho-1}(1-x)^{\rho}[1+ax+(1-b)]^{-2\rho-1}{}_2F_1\Big[\,\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\Big]\,\mathrm{d}x$$

$$= 2^{\alpha+\beta-2\rho}\ \frac{\Gamma(\rho-\frac{\alpha}{2}-\frac{\beta}{2})\Gamma(\frac{\alpha+\beta+2}{2})\Gamma(\rho)}{(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)}$$

$$\times\left[\frac{2\rho-\alpha+\beta)\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\rho-\frac{\alpha}{2}-1)\Gamma(\rho-\frac{\beta}{2}+\frac{1}{2})}-\frac{(2\rho-\alpha+\beta)\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\rho-\frac{\alpha}{2}+1)\Gamma(\rho-\frac{\beta}{2}+\frac{1}{2})}\right] \tag{2.1}$$

Where $Re(\rho)>0, Re(2\rho-\alpha-\beta)>0$, a and b are constants , such the expression

$1+ax+b(1-x)$ is not zero.

$$\int_0^1 x^{\rho-1}(1-x)^{\rho}\,[1+ax+b(1-x)]^{-2\rho+1}\,{}_2F_1\left[\,\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}\,\right]\mathrm{d}x$$

$$= 2^{\alpha+\beta-2\rho-1}\frac{\Gamma(\rho-\frac{\alpha}{2}-\frac{\beta}{2}-1)\Gamma(\frac{\alpha+\beta}{2})\Gamma(\rho-1)}{(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)}$$

$$\times\left[\frac{(2\rho-\alpha+\beta-2)\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\rho-\frac{\alpha}{2})\Gamma(\rho-\frac{\beta}{2}-\frac{1}{2})}+\frac{(2\rho+\alpha-\beta)\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})}{\Gamma(\rho-\frac{\beta}{2})\Gamma(\rho-\frac{\alpha}{2}-\frac{1}{2})}\right] \tag{2.2}$$

Where $Re(\rho)>0, Re(2\rho-\alpha-\beta)>0$, a and b are constants , such the expression

$1+ax+b(1-x)$ is not zero.

$$\int_0^{\pi/2} e^{i(2w+1)\pi\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}\ {}_2F_1\left[\,\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]\mathrm{d}\theta$$

$$=\frac{e^{i\pi(w+1)/2}\Gamma(w)\Gamma(w-\frac{\alpha'-\beta'}{2})\Gamma(\frac{\alpha'-\beta'}{2}+1)}{2^{2w-\alpha'-\beta'+2}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}$$

$$\times\left[\frac{(2w-\alpha'-\beta')\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})}{\Gamma(w-\frac{\alpha'}{2}+1)\Gamma(w-\frac{\beta'-1}{2})}-\frac{(2w+\alpha'-\beta')\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})}{\Gamma(w-\frac{\beta'}{2}+1)\Gamma(w-\frac{\alpha'-1}{2})}\right] \tag{2.3}$$

where $Re(w)>0$ and $Re(2w-\alpha'-\beta')>0$

$$\int_0^{\pi/2} e^{i(2w+1)\pi\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}\ {}_2F_1\Big[\,\alpha',\beta';\frac{\alpha'+\beta'}{2};e^{i\theta}cos\theta\Big]\,\mathrm{d}\theta$$

$$=\frac{e^{i\pi(w+1)/2}\Gamma(w-1)\Gamma(w-\frac{\alpha'-\beta'}{2}-1)\Gamma(\frac{\alpha'+\beta'}{2})}{2^{2w-\alpha'-\beta'}\Gamma(\alpha')\Gamma(\beta')}$$

$$\times\Big[\frac{(2w-\alpha'-\beta'-2)\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})}{\Gamma(w-\frac{\alpha'}{2})\Gamma(w-\frac{\beta'+1}{2})}-\frac{(2w+\alpha'-\beta'-2)\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})}{\Gamma(w-\frac{\beta'}{2})\Gamma(w-\frac{\alpha'+1}{2})}\Big] \tag{2.4}$$

where $Re(w) > 0$ and $Re(2w-\alpha'-\beta') > 0$

3 Finite double integrals

We evaluate the following four finite double integrals involving hypergeometric functions and Aleph-function.

$$\text{a) } \int_0^1 \int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+(1-b)]^{-2\rho-1} {}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$

$$e^{i(2w+1)\pi\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}\ {}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]$$

$$\times\aleph_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}\left(zx^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2i\theta w_1}(sin\theta)^{w_1}(cos\theta)^{w_1}\middle| \begin{matrix} \mathrm{A} \\ \mathrm{B} \end{matrix}\right) \mathrm{d}\theta\, \mathrm{d}x$$

$$= \frac{2^{\alpha+\beta-2\rho-2}\Gamma(\frac{\alpha+\beta+2}{2})}{\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\ \aleph_1(z) - \Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})\aleph_2(z)\right]$$

$$\times\frac{e^{i\pi(w+1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w-\alpha'-\beta'}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})\aleph_3(z) - \Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})\aleph_4(z)\right] \tag{3.1}$$

where

$$\aleph_1(z) = \aleph_{p_i+3,q_i+3,c_i;r}^{m,\mathfrak{n}+3}\left(\frac{z2^{-2\rho_1-2}}{(1+a)^{\rho_1}(1+b)^{\rho_1}}\middle| \begin{matrix} \mathrm{A}\ ,(\alpha-\beta-2\rho,2\rho_1),(1-\rho,\rho_1) \\ \mathrm{B}\ ,(1-2\rho+\alpha+\beta,2\rho_1),(\alpha/2-\rho,\rho_1)\end{matrix}\right.$$

$$\left.\begin{matrix},(1+(\alpha+\beta)/2-\rho,\rho_1) \\ ,((\beta+1)/2-\rho,\rho_1)\end{matrix}\right) \tag{3.2}$$

$$\aleph_2(z)= \aleph_{p_i+3,q_i+3,c_i;r}^{m,\mathfrak{n}+3}\left(\frac{z2^{-2\rho_1-2}}{(1+a)^{\rho_1}(1+b)^{\rho_1}}\middle| \begin{matrix} \mathrm{A}\ ,(-\alpha+\beta-2\rho,2\rho_1),(1-\rho,\rho_1) \\ \mathrm{B}\ ,(1-2\rho-\alpha+\beta,2\rho_1),((\alpha+1)/2-\rho,\rho_1)\end{matrix}\right.$$

$$\left.\begin{matrix},(1+(\alpha+\beta)/2-\rho,\rho_1) \\ ,(\beta/2-\rho,\rho_1)\end{matrix}\right) \tag{3.3}$$

$$\aleph_3(z) = \aleph_{p_i+3,q_i+3,c_i;r}^{m,\mathfrak{n}+3}\left(\frac{ze^{i\pi w_1/2}}{4^{w_1}}\middle| \begin{matrix} \mathrm{A}\ ,(\alpha'-\beta'-2w,w_1),(1-w,w_1) \\ \mathrm{B}\ ,(1-2\mathrm{w}+\alpha'+\beta',w_1),(\alpha'/2-w,w_1)\end{matrix}\right.$$

$$\left.\begin{matrix},(1+(\alpha'+\beta')/2-w,w_1) \\ ,((\beta'+1)/2-\rho,\rho_1)\end{matrix}\right) \tag{3.4}$$

$$\aleph_4(z)= \aleph_{p_i+3,q_i+3,c_i;r}^{m,\mathfrak{n}+3}\left(\frac{ze^{i\pi w_1/2}}{4^{w_1}}\middle| \begin{matrix} \mathrm{A}\ ,(-\alpha'+\beta'-2w,w_1),(1-w,w_1) \\ \mathrm{B}\ ,(1-2\mathrm{w}-\alpha'+\beta',w_1),((\alpha'+1)/2-w,w_1)\end{matrix}\right.$$

$$\left.\begin{matrix},(1+(\alpha'+\beta')/2-w,w_1) \\ ,(\beta'/2-w,w_1)\end{matrix}\right) \tag{3.5}$$

with the validity conditions : $Re(\rho) > 0, Re(w) > 0, |argz| < \frac{1}{2}\pi\Omega$,

$Re(2\rho - \alpha - \beta + 2\rho_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j}) > 0$ and $Re(2w - \alpha' - \beta' + 2w_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j}) > 0$

b) $$\int_0^1 \int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho+1} {}_2F_1\Big[\alpha, \beta; \frac{\alpha+\beta+2}{2}; \frac{x(1+a)}{1+ax+b(1-x)}\Big]$$

$$e^{i\pi(2w-1)\theta}(sin\theta)^{w-2}(cos\theta)^{w-1} \ {}_2F_1\Big[\alpha', \beta'; \frac{\alpha'+\beta'}{2}; e^{i\theta}cos\theta\Big]$$

$$\times \aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(zx^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2i\theta w_1}(sin\theta)^{w_1}(cos\theta)^{w_1}\middle| \begin{matrix} \mathrm{A} \\ \mathrm{B} \end{matrix}\right) \mathrm{d}\theta \, \mathrm{d}x$$

$$= \frac{2^{\alpha+\beta-2\rho-1}\Gamma(\frac{\alpha+\beta+2}{2})}{\Gamma(\alpha)\Gamma(\beta)(1+a)^{\rho}(1+b)^{\rho}} \left[\Gamma(\frac{\alpha+\beta}{2}) \ \Gamma(\frac{\beta}{2}) \ \aleph_5(z) - \Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+\beta}{2}) \ \aleph_6(z) \right]$$

$$\times \frac{e^{i\pi(w+1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w-\alpha'+1}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')} \left[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2}) \ \aleph_7(z) - \Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'+2}{2})\aleph_8(z) \right] \quad (3.6)$$

where

$$\aleph_5(z) = \aleph^{m,\mathfrak{n}+3}_{p_i+3,q_i+3,c_i;r}\left(\frac{2^{-2\rho_1}z}{(1+a)^{\rho_1}(1+b)^{\rho_1}} \middle| \begin{matrix} \mathrm{A}, (2+\alpha-\beta-2\rho, \rho_1), (2-\rho, \rho_1) \\ \mathrm{B}, (3-2\rho+\alpha+\beta, 2\rho_1), (1+\alpha/2-\rho, \rho_1) \end{matrix}\right.$$

$$\left. \begin{matrix} , (2+(\alpha+\beta)/2-\rho, \rho_1) \\ , ((\beta+3)/2-\rho, \rho_1) \end{matrix} \right) \quad (3.7)$$

$$\aleph_6(z) = \aleph^{m,\mathfrak{n}+3}_{p_i+3,q_i+3,c_i;r}\left(\frac{2^{-2\rho_1}z}{(1+a)^{\rho_1}(1+b)^{\rho_1}} \middle| \begin{matrix} \mathrm{A}, (2-\alpha+\beta-2\rho, \rho_1), (2-\rho, \rho_1) \\ \mathrm{B}, (3-2\rho-\alpha+\beta, 2\rho_1), (1+\beta/2-\rho, \rho_1) \end{matrix}\right.$$

$$\left. \begin{matrix} , (2+(\alpha+\beta)/2-\rho, \rho_1) \\ , ((\alpha+3)/2-\rho, \rho_1) \end{matrix} \right) \quad (3.8)$$

$$\aleph_7(z) = \aleph^{m,\mathfrak{n}+3}_{p_i+3,q_i+3,c_i;r}\left(\frac{ze^{i\pi w_1/2}}{4^{w_1}} \middle| \begin{matrix} \mathrm{A}, (2+\alpha'-\beta'-2w, w_1), (2-w, w_1) \\ \mathrm{B}, (3-2\mathrm{w}-\alpha'+\beta', 2w_1), (1+\alpha/2-w, w_1) \end{matrix}\right.$$

$$\left. \begin{matrix} , (2+(\alpha'+\beta')/2-w, w_1) \\ , ((\beta+3)/2-w, w_1) \end{matrix} \right) \quad (3.9)$$

$$\aleph_8(z) = \aleph^{m,\mathfrak{n}+3}_{p_i+3,q_i+3,c_i;r}\left(\frac{ze^{i\pi w_1/2}}{4^{w_1}} \middle| \begin{matrix} \mathrm{A}, (2-\alpha'+\beta'-2w, w_1), (2-w, w_1) \\ \mathrm{B}, (3-2\mathrm{w}-\alpha'+\beta', 2w_1), (1+\beta/2-w, w_1) \end{matrix}\right.$$

$$\left. \begin{matrix} , (2+(\alpha'+\beta')/2-w, w_1) \\ , ((\alpha+3)/2-w, w_1) \end{matrix} \right) \quad (3.10)$$

with the validity conditions : $Re(\rho) > 1, Re(w) > 1, |argz| < \frac{1}{2}\pi\Omega$, and

$Re(2\rho-\alpha-\beta+2\rho_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j})>2$ and $Re(2w-\alpha'-\beta'+2w_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j})>2$

c) $\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho+1}{}_2F_1\Big[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\Big]$

$\times e^{i\pi(2w-1)\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}\ {}_2F_1\Big[\alpha',\beta';\frac{\alpha'+\beta'}{2};e^{i\theta}cos\theta\Big]$

$\times\ \aleph_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}\Big(zx^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2i\theta w_1}(sin\theta)^{w_1}(cos\theta)^{w_1}\Big|\begin{matrix}\mathrm{A}\\ \mathrm{B}\end{matrix}\Big)\mathrm{d}\theta\,\mathrm{d}x$

$$=\frac{2^{\alpha+\beta-2\rho-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}}\Big[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\aleph_1(z)-\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})\,\aleph_2(z)\Big]$$

$$\times\,\frac{e^{i\pi(w-1)/2}\Gamma(\frac{\alpha'+\beta'}{2})}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')}\Big[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})\aleph_7(z)-\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})\,\aleph_8(z)\Big] \tag{3.11}$$

Where $\aleph_1(z),\aleph_2(z),\aleph_7(z)$ and $\aleph_8(z)$ are mentioned in (3.2) , (3.3) , (3.9) and (3.10) respectively an the validity conditions are the following :

$Re(\rho)>0,Re(w)>1,|argz|<\frac{1}{2}\pi\Omega$, and $Re(2\rho-\alpha-\beta+2\rho_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j})>0$

, $Re(2w-\alpha'-\beta'+2w_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j})>0$

d) $\int_0^1\int_0^{\pi/2} x^{\rho}(1-x)^{\rho-2}\ [1+ax+b(1-x)]^{-2\rho+1}{}_2F_1\Big[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\Big]$

$\times e^{i\pi(2w+1)\theta}(sin\theta)^{w}(cos\theta)^{w-1}\ {}_2F_1\Big[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\Big]$

$\times\ \aleph_{p_i,q_i,c_i;r}^{m,\mathfrak{n}}\Big(zx^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2i\theta w_1}(sin\theta)^{w_1}(cos\theta)^{w_1}\Big|\begin{matrix}\mathrm{A}\\ \mathrm{B}\end{matrix}\Big)\mathrm{d}\theta\,\mathrm{d}x$

$$=\frac{2^{\alpha+\beta-2\rho-1}\Gamma(\frac{\alpha+\beta}{2})\Gamma(\frac{\alpha+\beta+2}{2})}{\Gamma(\alpha)\Gamma(\beta)(1+a)^{\rho}(1+b)^{\rho}}\Big[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\,\aleph_5(z)-\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})\aleph_6(z)\Big]$$

$$\times\frac{e^{i\pi(w+1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')\Gamma(\alpha'-\beta')}\Big[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})\aleph_3(z)-\Gamma(\frac{\alpha'}{2})\Gamma(\frac{\beta'+1}{2})\aleph_4(z)\Big] \tag{3.12}$$

Where $\aleph_5(z),\aleph_6(z),\aleph_3(z)$ and $\aleph_4(z)$ are mentioned by (3.4) , (3.5) , (3.7) and (3.8) respectively and the validity conditions are :

$Re(\rho)>0,Re(w)>1,|argz|<\frac{1}{2}\pi\Omega$, and $Re(2\rho-\alpha-\beta+2\rho_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j})>0$

,$Re(2w-\alpha'-\beta'+2w_1 \min\limits_{1\leqslant j\leqslant n} \frac{b_j}{\beta_j})>2$

Proof : To etablish (3.1),we express the Aleph_function on the left hande side using (1.1) in Mellin-Barnes contour integral and interchanging the order of integration which is justifiable due to absolute convergence of the integrals , we have :

$$\frac{1}{2\pi\omega}\int_L \Omega^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}(s)\left(\left(\int_0^1 x^{\rho+\rho_1 s-1}(1-x)^{\rho+\rho_1 s}\left[1+ax+b(1-x)\right]^{-2\rho-2\rho_1 s-1}\right.\right.$$

$$\times\, {}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]\mathrm{d}x\Bigg)\left(\int_0^{\pi/2} e^{i(2w+2w_1 s+1)}\,(sin\theta)^{w+w_1 s}\right.$$

$$\left.\left.\times(cos\theta)^{w+w_1 s}\,{}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]\right)\mathrm{d}\theta\right)z^{-s}\mathrm{d}s$$

We evaluate the inner integrals with the help of (2.1) and (2.3) and applying (1.1) , we get the R.H.S of (3.1) in terms of product of Aleph-functions. The other integrals calculate in the similar method

4 Particular cases

If $a=b$ in (3.6) , we obtain :

$$\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}(1+b)^{-2\rho+1}\,{}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};x\right]e^{i\pi(2w+1)\theta}\,(sin\theta)^{w-2}$$

$$\times(cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}cos\theta\right]$$

$$\times\aleph^{m,\mathfrak{n}}_{p_i,q_i,c_i;r}\left(zx^{\rho_1}(1-x)^{\rho_1}(1+b)^{-2\rho_1}e^{2iw_1\theta}\,(sin\theta)^{w_1}(cos\theta)^{w_1}\,\middle|\,\begin{matrix}\mathrm{A}\\ \mathrm{B}\end{matrix}\right)\mathrm{d}\theta\,\mathrm{d}x$$

$$=\frac{2^{\alpha+\beta-2\rho_1-1}\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(\alpha)\Gamma(\beta)(1+b)^{2\rho}}\left[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta}{2})\aleph_5(z)-\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta+1}{2})\,\aleph_6(z)\right]$$

$$\times\frac{e^{i\pi(w_1-1)/2}\Gamma(\frac{\alpha'+\beta'+2}{2})}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma(\frac{\alpha'+1}{2})\Gamma(\frac{\beta'}{2})\,\aleph_7(z)-\Gamma(\frac{\alpha'+1}{2})\,\Gamma(\frac{\beta'+2}{2})\,\aleph_8(z)\right]\tag{4.1}$$

Remarks : if $c_i=1$ for $i=1,\cdots,r$, the Aleph-function degenere into the I_function defined by V.P. Saxena [6], for more details see D.kumar et al [1,5]. If r = 1 ,the I_function degenere into the fox's H-function , see Ronghe [4].

5 Conclusion

The aleph-function, presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as I-function ,Fox's H-function see [2] , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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