A double integral involving Aleph-function of several variables

$F.Y.\ AYANT^{1}$

1 Teacher in High School , France

ABSTRACT
In this document, we obtain an double integral involving the multivariable Aleph-function, the general class of polynomials of several variables and Aleph-function of one variable which are the sufficiently general in nature and are capable of yielding a large number of simpler and useful results merely by specializing the parameters in them. Further we establish some special cases.

KEYWORDS : Aleph-function of several variables, integrals, general class of polynomials,,Aleph-function of one variable

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

1.Introduction and preliminaries.

The Aleph- function , introduced by Südland [7] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i,Q_i,c_i;r}^{M,N}\left(z \;\middle|\; \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right) = \frac{1}{2\pi\omega}\int_L \Omega_{P_i,Q_i,c_i;r}^{M,N}(s) z^{-s} \mathrm{d}s \quad (1.1)$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r}^{M,N}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j + B_j s)\prod_{j=1}^{N}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r} c_i \prod_{j=N+1}^{P_i}\Gamma(a_{ji} + A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1 - b_{ji} - B_{ji}s)} \quad (1.2)$$

With $|argz| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0, i = 1, \cdots, r$

For convergence conditions and other details of Aleph-function , see Südland et al [7].The serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph_{P_i,Q_i,c_i;r}^{M,N}(z) = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g \Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_G g!} z^{-s} \quad (1.3)$$

With $s = \eta_{G,g} = \frac{b_G + g}{B_G}, P_i < Q_i, |z| < 1$ and $\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)$ is given in (1.2) (1.4)

The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1, K_1; \cdots; N_s, K_s] y_1^{K_1}\cdots y_s^{K_s} \quad (1.5)$$

Where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_s, K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!} A[N_1, K_1; \cdots ; N_s, K_s] \text{and} \quad U_{32} = p_i + 3; q_i + 2; \tau_i; R \quad (1.6)$$

The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [4] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

$$\text{We have : } \aleph(z_1, \cdots, z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;\mathfrak{R}:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c|c} z_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & \end{array} \right.$$

$$[(\mathrm{a}_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}] \quad , [\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] :$$
$$\cdots\cdots\cdots \quad , [\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}] :$$

$$\left. \begin{array}{l} [(\mathrm{c}_j^{(1)}), \gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}]; \cdots ; \ ; [(\mathrm{c}_j^{(r)}), \gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}), \delta_j^{(1)})_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}]; \cdots ;; [(\mathrm{d}_j^{(r)}), \delta_j^{(r)})_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \quad (1.7)$$

with $\omega = \sqrt{-1}$

For more details, see Ayant [1].

The reals numbers τ_i are positives for $i = 1, \cdots, \mathfrak{R}$, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1, \cdots, R^{(k)}$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|argz_k| < \frac{1}{2} A_i^{(k)} \pi \ , \ \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)} \quad (1.8)$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1} \ldots |z_r|^{\alpha_r}) , max(|z_1| \ldots |z_r|) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1} \ldots |z_r|^{\beta_r}) , min(|z_1| \ldots |z_r|) \rightarrow \infty$$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; \mathfrak{R} \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.9}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots , p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.10}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.11}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.12}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots , \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.13}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.14}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_r) = \aleph_{U:W}^{0,\mathrm{n}:V} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} \mathrm{A} : \mathrm{C} \\ \cdots \\ \mathrm{B} : \mathrm{D} \end{array} \right) \tag{1.15}$$

2. Required integrals

We have the two following integrals :

$$1) \int_0^{\pi/2} e^{i(\alpha+\beta)\theta} (sin\theta)^{\alpha-1} (cos\theta)^{\beta-1} \mathrm{d}\theta = e^{i\pi\alpha/2} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, Re(\alpha) > 0, Re(\beta) > 0 \tag{2.1}$$

$$2) \int_0^{\infty} x^{r-1/2} \left[(x+a)(x+b)\right]^{-r} \mathrm{d}x = \sqrt{\pi}(\sqrt{a}+\sqrt{b})^{1-2r} \frac{\Gamma\left(r-\frac{1}{2}\right)}{\Gamma(r)}, Re(r) > \frac{1}{2} \tag{2.2}$$

3. Main integral

Let $g(r, \theta, \alpha, \beta, \gamma) = e^{i(\alpha+\beta)\theta} (sin\theta)^{\alpha} (cos\theta)^{\beta} \left[\frac{r(\sqrt{a}+\sqrt{b})^2}{(r+a)(r+b)} \right]^{\gamma}$ and $U_{32} = p_i + 3, q_i + 2, \tau_i; \mathfrak{R}$ (3.1)

In this section, we will evaluate the general double integral with the helps of results (2.1) and (2.2). We have the general relation:

$$\int_0^{\infty} \int_0^{\pi/2} \frac{g(r,\theta,\alpha,\beta,\gamma)}{\sqrt{r} sin\theta cos\theta} \aleph_{P_i,Q_i,c_i;r'}^{M,N}(zg(r,\theta,c,d,e)) S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s} \left(\begin{array}{c} \mathrm{y}_1 g[r,\theta,c_1,d_1:e_1] \\ \cdots \\ \mathrm{y}_s g[r,\theta,c_s,d_s:e_s] \end{array} \right)$$

$$\aleph_{U:W}^{0,\mathrm{n}:V} \left(\begin{array}{c} \mathrm{z}_1 g[r,\theta,\delta_1,\mu_1:\lambda_1] \\ \cdots \\ \mathrm{z}_R g[r,\theta,\delta_R,\mu_R:\lambda_R] \end{array} \right) \mathrm{d}r\mathrm{d}\theta = \sqrt{\pi} e^{i\pi\alpha/2} (\sqrt{a}+\sqrt{b}) \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} a_1$$

$$\frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!} z^{\eta_{G,g}} y_1^{K_1} \cdots y_s^{K_s} e^{i\pi/2(c\eta_{G,g}+\sum_{i=1}^s K_i c_i)} (\sqrt{a}+\sqrt{b})^{-2(e\eta_{G,g}+\sum_{i=1}^s K_i e_i)}$$

$$\aleph_{U_{32}:W}^{0,\mathfrak{n}+3:V}\left(\begin{array}{c} \mathrm{z}_1\frac{e^{i\pi\delta_1/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_1}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_R\frac{e^{i\pi\delta_R/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_R}} \end{array}\middle| \begin{array}{c} (\frac{3}{2}\text{-}\gamma-e\eta_{G,g}-\sum_{i=1}^s K_ie_i;\lambda_1,\cdots,\lambda_R), \\ \cdot\ \cdot\ \cdot \\ (1\text{-}\gamma-e\eta_{G,g}-\sum_{i=1}^s K_ie_i;\lambda_1,\cdots,\lambda_R), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\alpha-d\eta_{G,g}-\sum_{i=1}^s K_id_i;\mu_1,\cdots,\mu_R),(1-\beta-c\eta_{G,g}-\sum_{i=1}^s K_ic_i;\delta_1,\cdots,\delta_R),A:C \\ \cdot\ \cdot\ \cdot \\ (1\text{-}\alpha-\beta-(c+d)\eta_{G,g}-\sum_{i=1}^s K_i(c_i+d_i);\mu_1+\delta_1,\cdots,\mu_R+\delta_R),B:D \end{array}\right) \quad (3.2)$$

Provided that

a) $min(c,d,e,c_i,d_,e_i,\delta_j,\mu_j,\lambda_j)>0, i=1,\cdots,s; j=1,\cdots,R$

b) $Re[\beta+c\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^R\mu_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $Re[\alpha+d\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^R\delta_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $Re[\gamma+e\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^R\lambda_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>\frac{1}{2}$

e) $Re[\alpha+d\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\delta_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0, i=1,\cdots,R$

f) $Re[\beta+c\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\mu_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0, i=1,\cdots,R$

g) $Re[\gamma+e\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\lambda_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>\frac{1}{2}, i=1,\cdots,R$

h) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.8)

i) $|argz|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^M\beta_j+\sum_{j=1}^N\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

Proof of (3.2). Let $M=\frac{1}{(2\pi\omega)^R}\int_{L_1}\cdots\int_{L_R}\psi(s_1,\cdots,s_R)\prod_{k=1}^R\theta_k(s_k)z_k^{s_k}$

To obtain (3.2), express a general class of polynomials of several variables occurring in the integrand of (3.2) as defined in (1.5),series representation of the Aleph-function by (1.3) and the multivariable Aleph-function by its Mellin-Barnes contour integral with the help of (1.7). Now we interchange the order of summation and integrations (which is permissible under the conditions stated above), we obtain :

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\int_0^{\infty}\int_0^{\pi/2}\frac{g(r,\theta,\alpha,\beta,\gamma)}{\sqrt{r}sin\theta cos\theta}$$

$$(g(r,\theta,c,d,e))^{\eta_{G,g}}\prod_{i=1}^{s}(g(r,\theta,c_i,d_i,e_i))^{K_i}\ M\left\{\prod_{k=1}^{R}(g(r,\theta,\delta_k,\mu_k,\lambda_k))^{s_k}\right\}\mathrm{d}s_1\cdots\mathrm{d}s_R\mathrm{d}r\mathrm{d}\theta \qquad (3.3)$$

Assuming the inversion of order of integrations in (3.2) to be permissible by absolute convergence of the integrals involved, we have :

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}$$

$$M\left(\left(\int_0^{\infty}\left[\frac{r(\sqrt{a}+\sqrt{b})^2}{(r+a)(r+b)}\right]^{\gamma+e\eta_{G,g}+\sum_{i=1}^{s}K_ie_i+\sum_{k=1}^{R}\lambda_ks_k}\mathrm{d}r\right)\right.$$

$$\left(\int_0^{\pi/2}e^{i(\alpha+\beta+(c+d)\eta_{G,g}+\sum_{i=1}^{s}K_i(c_i+d_i)+\sum_{k=1}^{R}s_i(\delta_i+\mu_i))\theta}\left(cos\theta\right)^{\alpha+c\eta_{G,g}+\sum_{i=1}^{s}K_ic_i+\sum_{k=1}^{R}\delta_is_i}\right.$$

$$\left.\left(sin\theta\right)^{\beta+d\eta_{G,g}+\sum_{i=1}^{s}K_id_i+\sum_{k=1}^{R}\mu_is_i}\mathrm{d}\theta\right)\mathrm{d}s_1\cdots\mathrm{d}s_R \qquad (3.4)$$

We evaluate the inner integrals with the help of (2.1) and (2.2), we get

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\sqrt{\pi}e^{i\pi\alpha/2}(\sqrt{a}+\sqrt{b})$$

$$\left(\sqrt{a}+\sqrt{b}\right)^{-2(e\eta_{G,g}+\sum_{i=1}^{s}K_ie_i+\sum_{k=1}^{R}s_k\lambda_i)}e^{i\pi/2(\eta_{G,g}c+\sum_{i=1}^{s}K_ic_i+\sum_{k=1}^{R}s_k\delta_k)}$$

$$M\left(\frac{\Gamma(\alpha+c\eta_{G,g}+\sum_{i=1}^{s}K_ic_i+\sum_{k=1}^{R}\delta_ks_k)\Gamma(\beta+d\eta_{G,g}+\sum_{i=1}^{s}K_id_i+\sum_{k=1}^{R}\mu_ks_k)}{\Gamma(\alpha+\beta+(c+d)\eta_{G,g}+\sum_{i=1}^{s}K_i(c_i+d_i)+\sum_{k=1}^{R}(\delta_k+\mu_k)s_k)}\right.$$

$$\left.\frac{\Gamma(\gamma+e\eta_{G,g}+\sum_{i=1}^{s}K_is_i+\sum_{k=1}^{R}s_k\lambda_k+\frac{1}{2})}{\Gamma(\gamma+e\eta_{G,g}+\sum_{i=1}^{s}K_is_i+\sum_{k=1}^{R}s_k\lambda_k)}\right)\mathrm{d}s_1\cdots\mathrm{d}s_R \qquad (3.5)$$

Finally interpreting the resulting Mellin-Barnes contour integral as a multivariable Aleph-function, we obtain the desired result (3.2).

4. Multivariable I-function

If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(R)}}\rightarrow 1$, the Aleph-function of several variables degenere to the I-function of several variables. The general double integral have been derived in this section for multivariable I-functions defined by Sharma et al [3].

$$\int_0^\infty \int_0^{\pi/2} \frac{g(r,\theta,\alpha,\beta,\gamma)}{\sqrt{r}sin\theta cos\theta} \aleph_{P_i,Q_i,c_i;r'}^{M,N}\big(zg(r,\theta,c,d,e)\big) S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1 g[r,\theta,c_1,d_1:e_1] \\ \cdots \\ \mathrm{y}_s g[r,\theta,c_s,d_s:e_s] \end{array}\right)$$

$$I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1 g[r,\theta,\delta_1,\mu_1:\lambda_1] \\ \cdots \\ \mathrm{z}_R g[R,\theta,\delta_R,\mu_R:\lambda_R] \end{array}\right) \mathrm{d}r\mathrm{d}\theta = \sqrt{\pi}e^{i\pi\alpha/2}(\sqrt{a}+\sqrt{b}) \sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1$$

$$\frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!} z^{\eta_{G,g}} y_1^{K_1}\cdots y_s^{K_s} e^{1\pi\alpha/2(\eta_{G,g}+\sum_{i=1}^s K_i)} \big(\sqrt{a}+\sqrt{b}\big)^{-2(\eta_{G,g}+\sum_{i=1}^s K_i)}$$

$$I_{U_{32}:W}^{0,\mathfrak{n}+3:V}\left(\begin{array}{c} \mathrm{z}_1 \frac{e^{i\pi\delta_1/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_R \frac{e^{i\pi\delta_R/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_R}} \end{array}\middle| \begin{array}{c} (\frac{3}{2}\text{-}\gamma - e\eta_{G,g} - \sum_{i=1}^s K_i e_i;\lambda_1,\cdots,\lambda_R), \\ \cdots \\ (1\text{-}\gamma - e\eta_{G,g} - \sum_{i=1}^s K_i e_i;\lambda_1,\cdots,\lambda_R), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\alpha - d\eta_{G,g} - \sum_{i=1}^s K_i d_i;\mu_1,\cdots,\mu_R),(1-\beta-c\eta_{G,g}-\sum_{i=1}^s K_i c_i;\delta_1,\cdots,\delta_R),A:C \\ \cdots \\ (1\text{-}\alpha-\beta-(c+d)\eta_{G,g}-\sum_{i=1}^s K_i(c_i+d_i);\mu_1+\delta_1,\cdots,\mu_R+\delta_R),B:D \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.2)

5. Aleph-function of two variables

If $R = 2$, we obtain the Aleph-function of two variables defined by K.Sharma [5], and we have the following double integral.

$$\int_0^\infty \int_0^{\pi/2} \frac{g(r,\theta,\alpha,\beta,\gamma)}{\sqrt{r}sin\theta cos\theta} \aleph_{P_i,Q_i,c_i;r'}^{M,N}\big(zg(r,\theta,c,d,e)\big) S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1 g[r,\theta,c_1,d_1:e_1] \\ \cdots \\ \mathrm{y}_s g[r,\theta,c_s,d_s:e_s] \end{array}\right)$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1 g[r,\theta,\delta_1,\mu_1:\lambda_1] \\ \cdots \\ \mathrm{z}_2 g[r,\theta,\delta_2,\mu_2:\lambda_2] \end{array}\right) \mathrm{d}r\mathrm{d}\theta = \sqrt{\pi}e^{i\pi\alpha/2}(\sqrt{a}+\sqrt{b}) \sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1$$

$$\frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!} z^{\eta_{G,g}} y_1^{K_1}\cdots y_s^{K_s} e^{1\pi\alpha/2(\eta_{G,g}+\sum_{i=1}^s K_i)} \big(\sqrt{a}+\sqrt{b}\big)^{-2(\eta_{G,g}+\sum_{i=1}^s K_i)}$$

$$\aleph_{U_{32}:W}^{0,\mathfrak{n}+3:V}\left(\begin{array}{c} \mathrm{z}_1 \frac{e^{i\pi\delta_1/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_2 \frac{e^{i\pi\delta_2/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_2}} \end{array}\middle| \begin{array}{c} (\frac{3}{2}\text{-}\gamma - e\eta_{G,g} - \sum_{i=1}^s K_i e_i;\lambda_1,\lambda_2), \\ \cdots \\ (1\text{-}\gamma - e\eta_{G,g} - \sum_{i=1}^s K_i e_i;\lambda_1,\lambda_2), \end{array}\right.$$

$$\left.\begin{matrix}(1\text{-}\alpha - d\eta_{G,g} - \sum_{i=1}^{s} K_i d_i;\mu_1,\mu_2),(1-\beta-c\eta_{G,g}-\sum_{i=1}^{s}K_ic_i;\delta_1,\delta_2),A:C \\ \cdots \\ (1\text{-}\alpha-\beta-(c+d)\eta_{G,g}-\sum_{i=1}^{s}K_i(c_i+d_i);\mu_1+\delta_1,\mu_2+\delta_2),B:D\end{matrix}\right) \quad (5.1)$$

under the same conditions and notations that (3.2)

6. I-function of two variables

If $\tau_i,\tau_i',\tau_i''\rightarrow 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [4] and we obtain the same formulae with the I-function of two variables

$$\int_0^\infty\int_0^{\pi/2}\frac{g(r,\theta,\alpha,\beta,\gamma)}{\sqrt{r}sin\theta cos\theta}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\big(zg(r,\theta,c,d,e)\big)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{y}_1g[r,\theta,c_1,d_1:e_1]\\ \cdots \\ \mathrm{y}_sg[r,\theta,c_s,d_s:e_s]\end{pmatrix}$$

$$I_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix}\mathrm{z}_1g[r,\theta,\delta_1,\mu_1:\lambda_1]\\ \cdots \\ \mathrm{z}_2g[r,\theta,\delta_2,\mu_2:\lambda_2]\end{pmatrix}\mathrm{d}r\mathrm{d}\theta=\sqrt{\pi}e^{i\pi\alpha/2}(\sqrt{a}+\sqrt{b})\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}a_1$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_Gg!}z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}e^{1\pi\alpha/2(\eta_{G,g}+\sum_{i=1}^{s}K_i)}\big(\sqrt{a}+\sqrt{b}\big)^{-2(\eta_{G,g}+\sum_{i=1}^{s}K_i)}$$

$$I_{U_{32}:W}^{0,\mathfrak{n}+3:V}\left(\begin{matrix}\mathrm{z}_1\frac{e^{i\pi\delta_1/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_1}}\\ \cdots \\ \cdots \\ \mathrm{z}_2\frac{e^{i\pi\delta_2/2}}{(\sqrt{a}+\sqrt{b})^{2\lambda_2}}\end{matrix}\ \middle|\ \begin{matrix}(\frac{3}{2}\text{-}\gamma-e\eta_{G,g}-\sum_{i=1}^{s}K_ie_i;\lambda_1,\lambda_2),\\ \cdots \\ (1\text{-}\gamma-e\eta_{G,g}-\sum_{i=1}^{s}K_ie_i;\lambda_1,\lambda_2),\end{matrix}\right.$$

$$\left.\begin{matrix}(1\text{-}\alpha - d\eta_{G,g} - \sum_{i=1}^{s} K_i d_i;\mu_1,\mu_2),(1-\beta-c\eta_{G,g}-\sum_{i=1}^{s}K_ic_i;\delta_1,\delta_2),A:C \\ \cdots \\ (1\text{-}\alpha-\beta-(c+d)\eta_{G,g}-\sum_{i=1}^{s}K_i(c_i+d_i);\mu_1+\delta_1,\mu_2+\delta_2),B:D\end{matrix}\right) \quad (5.1)$$

under the same conditions and notations that (3.2)

7. Conclusion

Due to general nature of the multivariable aleph-function and the double integral involving here, our formulas are capable to be reduced into many known and news integrals involving the special functions of one and several variables and polynomials of one and several variables.

REFERENCES

[1] Ayant F.Y. An integral associated with the Aleph-functions of several variables. *International Journal of Mathematics Trends and Technology (IJMTT)*. 2016 Vol 31 (3), page 142-154.

[2] Chaurasia V.B.L and Singh Y. New generalization of integral equations of fredholm type using Aleph-function Int. J. of Modern Math. Sci. 9(3), 2014, p 208-220.

[3] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[4] C.K. Sharma and P.L. mishra : On the I-function of two variables and its properties. Acta Ciencia Indica Math , 1991 Vol 17 page 667-672.

[5] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page1-13.

[6] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.

[7] Südland N.; Baumann, B. and Nonnenmacher T.F. , Open problem : who knows about the Aleph-functions? Fract. Calc. Appl. Anal., 1(4) (1998): 401-402.

Personal adress : 411 Avenue Joseph Raynaud

Le parc Fleuri , Bat B

83140 , Six-Fours les plages

Tel : 06-83-12-49-68

Department : VAR

Country : FRANCE