Evaluation of certain finite integrals

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ABSTRACT
In this paper, we evaluate two finite integrals involving the product of a generalized hypergeometric function, general class of multivariable polynomials, Aleph-function of one variable and generalized multivariable Aleph-function.The mains results of our document are quite general in nature and capable of yielding a very large number of integrals involving polynomials and various special functions occuring in the problem of mathematical analysis and mathematical physics and mechanics.

Keywords :generalized multivariable Aleph-function, Aleph-function, class of multivariable polynomials, generalized hypergeometric function, finite integral,.

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1. Introduction and preliminaries.

The Aleph- function , introduced by Südland [7] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(z\ \middle|\ \begin{matrix}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)z^{-s}\mathrm{d}s \tag{1.1}$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.2}$$

With :

$$|argz|<\frac{1}{2}\pi\Omega\quad \text{Where } \Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0\quad \text{with } i=1,\cdots,r$$

For convergence conditions and other details of Aleph-function , see Südland et al [7].

Serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph_{P_i,Q_i,c_i;r}^{M,N}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_Gg!}z^{-s} \tag{1.3}$$

With $s=\eta_{G,g}=\dfrac{b_G+g}{B_G}, P_i<Q_i, |z|<1$ and $\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)$ is given in (1.2) (1.4)

The generalized polynomials of multivariables defined by Srivastava [5], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s]=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.6}$$

where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients are $A[N_1,K_1;\cdots;N_s,K_s]$ arbitrary constants, real or complex.

The generalized hypergeometric function serie is defined as follows :

$$_{p'}F_{q'}(y)=\sum_{s'=0}^{\infty}\frac{[(a_{p'})]_{s'}}{[(b_{q'})]_{s'}}y^{s'} \tag{1.7}$$

Here $[(a_{p'})]_{s'} = (a_1)_{s'}\cdots(a_{p'})_{s'}\,;[(b_{q'})]_{s'} = (b_1)_{s'}\cdots(b_{q'})_{s'}$.
The serie (1.7) converge if $p' \leqslant q' and\, |y| < 1$.

In the document , we note : $a' = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s]$ (1.8)

The generalized Aleph-function of several variables generalize the multivariable I-function defined by H.M. Sharma and Ahmad [3] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r) = \aleph^{m,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$[(\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})_{1,m}],[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$

$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.9}$$

with $\omega = \sqrt{-1}$

For more details, see Ayant [1].
The reals numbers τ_i are positives for $i = 1,\cdots,R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1,\cdots,R^{(k)}$
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\sum_{j=1}^{m}\beta_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=m+1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0,\ \text{with}\ k = 1\cdots,r,\, i = 1,\cdots,R\ ,i^{(k)} = 1,\cdots,R^{(k)} \tag{1.10}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1}\ldots|z_r|^{\alpha_r}\,), max(\,|z_1|\ldots|z_r|\,)\rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1}\ldots|z_r|^{\beta_r}\,), min(\,|z_1|\ldots|z_r|\,)\rightarrow\infty$

where, with $k = 1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.11}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.12}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.13}$$

$$B = \{(b_j; \beta_j, \cdots, \beta_j)_{1,m}\}, \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.14}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.15}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.16}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{m,n:V}\left(\begin{array}{c|c} z_1 & A : C \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ \cdot & \\ z_r & B : D \end{array}\right) \tag{1.17}$$

2. Main integrals

In this section, we shall evaluate the following two finite integrals. The integrals are associated with the product of a generalized hypergeometric function, general class of multivariable polynomials, Aleph-function of one variable and multivariable Aleph-function.

First integral

$$\int_a^b (x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}{}_{p'}F_{q'}\left((a_{p'}); (b_{q'}); w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\frac{(x-a)^u(b-x)^v}{(cx+d)^p(gx+f)^q}\right) S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1\frac{(x-a)^{\sigma_1}(b-x)^{\eta_1}}{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ y_s\frac{(x-a)^{\sigma_s}(b-x)^{\eta_s}}{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}} \end{array}\right) \aleph_{U:W}^{m,n:V}\left(\begin{array}{c} z_1\frac{(x-a)^{u_1}(b-x)^{v_1}}{(cx+d)^{p_1}(gx+f)^{q_1}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ z_r\frac{(x-a)^{u_r}(b-x)^{v_r}}{(cx+d)^{p_r}(gx+f)^{q_r}} \end{array}\right) dx$$

$$= (b-a)^{\lambda+\mu-1}(ac+d)^{\gamma}(bg+f)^{\delta}\sum_{l_1,l_2,s'=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{a'b'g'(\eta_{G,g})f(s')}{l_1!l_2!s'!}$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\left(\frac{c(a-b)}{ac+d}\right)^{l_1}\left(\frac{g(b-a)}{bg+f}\right)^{l_2}\aleph_{U_{43}:W}^{m,n+4:V}\left(\begin{array}{c|} z_1\frac{(b-a)^{u_1+v_1}}{(ac+d)^{p_1}(bg+f)^{q_1}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ z_r\frac{(b-a)^{u_r+v_r}}{(ac+d)^{p_r}(bg+f)^{q_r}} \end{array}\right.$$

$$(1\text{-}\lambda - u\eta_{G,g} - es' - l_1 - \textstyle\sum_{i=1}^s \sigma_i K_i : u_1, \cdots, u_r),$$

$$\cdot\ \cdot\ \cdot$$

$$(1+\gamma - p\eta_{G,g} - e's' - \textstyle\sum_{i=1}^s \lambda_i K_i : p_1, \cdots, p_r),$$

$(1\text{-}\mu - v\eta_{G,g} - hs' - l_2 - \sum_{i=1}^{s}\eta_i K_i : v_1, \cdots, v_r),$

$\cdots$

$(1{+}\delta - q\eta_{G,g} - h's' - \sum_{i=1}^{s}\mu_i K_i : q_1, \cdots, q_r),$

$(1{+}\gamma - p\eta_{G,g} - e's' - l_1 - \sum_{i=1}^{s}\lambda_i K_i : p_1, \cdots, p_r),$

$\cdots$

$(1\text{-}\lambda - \mu - (u+v)\eta_{G,g} - (e+h)s' - l_1 - l_2 - \sum_{i=1}^{s}(\sigma_i + \eta_i)K_i : u_1 + v_1, \cdots, u_r + v_r),$

$$\left.\begin{matrix}(1{+}\delta - q\eta_{G,g} - h's' - l_2 - \sum_{i=1}^{s}\mu_i K_i : q_1, \cdots, q_r), A : B \\ \cdots \\ \mathrm{B : D}\end{matrix}\right) \tag{2.1}$$

where a' is defined by (1.8), $U_{43} = p_i + 4, q_i + 3, \tau_i; R$

$$f(s') = \frac{\prod_{i=1}^{p'}(a_i)_{s'}}{\prod_{i=1}^{q'}(b_i)_{s'}}\left\{\frac{w(b-a)^{h+e}}{(ac+d)^{e'}(bg+f)^{h'}}\right\}^{s'} \tag{2.2}$$

$$g'(\eta_{G,g}) = \left\{\frac{z(b-a)^{u+v}}{(ac+d)^p(bg+f)^q}\right\}^{\eta_{G,g}} \tag{2.3}$$

$$b' = \prod_{i=1}^{s}\left\{\frac{y_i^{K_i}(b-a)^{(\sigma_i+\eta_i)K_i}}{(ac+d)^{\lambda_i K_i}(bg+f)^{\mu_i K_i}}\right\} \tag{2.4}$$

Provided that

a) $Re(\lambda, \mu, e, h, e', h', p, q, u, v) > 0$

b) $min(u_i, v_i, p_i, q_i) \geqslant 0, i = 1, \cdots, r$ (not all zero simultaneously)

c) When $min(\sigma_i, \eta_i) \geqslant 0, i = 1, \cdots, s$, we have

i) $Re[\lambda + u \min_{1 \leqslant j \leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{r} u_i \min_{1 \leqslant j \leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

ii) $Re[\mu + v \min_{1 \leqslant j \leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{r} v_i \min_{1 \leqslant j \leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

When $max(\sigma_i, \eta_i) < 0$, $i = 1, \cdots, s$, we have

iii) $Re[\lambda + u \min_{1 \leqslant j \leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}\left[\sigma_i \frac{N_i}{M_i}\right] + \sum_{i=1}^{r} u_i \min_{1 \leqslant j \leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

iV) $Re[\mu + v \min_{1 \leqslant j \leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}\left[\eta_i \frac{N_i}{M_i}\right] + \sum_{i=1}^{r} v_i \min_{1 \leqslant j \leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

When $\sigma_i \geqslant 0, \eta_i < 0, i = 1, \cdots, s$, the conditions i) and iV) are satisfied

When $\eta_i \geqslant 0, \sigma_i < 0, i = 1, \cdots, s$, the conditions ii) and iii) are satisfied

d) $max\left\{\left|\frac{c(b-a)}{ac+d}\right|, \left|\frac{g(b-a)}{bg+f}\right|\right\} < 1, b \neq a$

e) $p' \leqslant q'$ or $p' = q' + 1$ and $\left|\frac{w(b-a)^{h+e}}{(ac+d)^{e'}(bg+f)^{h'}}\right| < 1$

f) $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.10)

g) $|argz| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

Second integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}}{(cx+d)^{\gamma}(gx+f)^{\delta}} {}_{p'}F_{q'}\left((a_{p'});(b_{q'});w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\right)\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\frac{(cx+d)^p(gx+f)^q}{(x-a)^u(b-x)^v}\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1\frac{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}}{(x-a)^{\sigma_1}(b-x)^{\eta_1}} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_s\frac{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}}{(x-a)^{\sigma_s}(b-x)^{\eta_s}} \end{array}\right)\aleph_{U:W}^{m,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1\frac{(cx+d)^{p_1}(gx+f)^{q_1}}{(x-a)^{u_1}(b-x)^{v_1}} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r\frac{(cx+d)^{p_r}(gx+f)^{q_r}}{(x-a)^{u_r}(b-x)^{v_r}} \end{array}\right)\mathrm{d}x$$

$$= \frac{(b-a)^{\lambda+\mu-1}}{(ac+d)^{\gamma}(bg+f)^{\delta}}\sum_{l_1,l_2,s'=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{a'b''g'(\eta_{G,g})f(s')(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{l_1!l_2!s'!\qquad B_G g!}$$

$$\left(\frac{c(a-b)}{ac+d}\right)^{l_1}\left(\frac{g(b-a)}{bg+f}\right)^{l_2}\aleph_{U_{34}:W}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1\frac{(ac+d)^{p_1}(bg+f)^{q_1}}{(b-a)^{u_1+v_1}} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r\frac{(ac+d)^{p_r}(bg+f)^{q_r}}{(b-a)^{u_r+v_r}} \end{array}\right|$$

$$(\gamma - p\eta_{G,g} + e's' - \textstyle\sum_{i=1}^{s}\lambda_i K_i : p_1,\cdots,p_r), \qquad (\delta - q\eta_{G,g} + h's' - \textstyle\sum_{i=1}^{s}\mu_i K_i : q_1,\cdots,q_r),$$

$$\cdot\cdot\cdot \qquad\qquad \cdot\cdot\cdot$$

$$(\gamma + l_1 + e's' - p\eta_{G,g} - \textstyle\sum_{i=1}^{s}\lambda_i K_i : p_1,\cdots,p_r),(\delta + l_2 - q\eta_{G,g} + h's' - \textstyle\sum_{i=1}^{s}\mu_i K_i : q_1,\cdots,q_r),$$

$$(\lambda + \mu + l_1 + l_2 + (e+h)s' - (u+v)\eta_{G,g} - \textstyle\sum_{i=1}^{s}(\sigma_i + \eta_i)K_i; u_1 + v_1,\cdots,u_r + v_r),$$

$$\cdot\cdot\cdot$$

$$(\lambda + l_1 + es' - u\eta_{G,g} - \textstyle\sum_{i=1}^{s}\sigma_i K_i : u_1,\cdots,u_r),$$

$$\left.\begin{array}{c} \mathrm{A}:\mathrm{C} \\ \cdot\cdot\cdot \\ (\mu + l_2 + hs' - v\eta_{G,g} - \sum_{i=1}^{s}\eta_i K_i : v_1,\cdots,v_r), C : D \end{array}\right) \tag{2.5}$$

where a' is defined by (1.8), $U_{34} = p_i + 3, q_i + 4, \tau_i; R, f(s')$ is defined by (2.2)

$$g'(\eta_{G,g}) = \left\{ \frac{(ac+d)^p(bg+f)^q}{z(b-a)^{u+v}} \right\}^{\eta_{G,g}} \tag{2.6}$$

$$b'' = \prod_{i=1}^{s} \left\{ \frac{(ac+d)^{\lambda_i K_i}(bg+f)^{\mu_i K_i}}{y_i^{K_i}(b-a)^{(\sigma_i+\eta_i)K_i}} \right\} \tag{2.7}$$

Provided that

a) $Re(\lambda, \mu, e, h, e', h', p, q, u, v) > 0$

b) $min(u_i, v_i, p_i, q_i) \geqslant 0, i = 1, \cdots, r$ (not all zero simultaneously)

c) When $min(\sigma_i, \eta_i) \geqslant 0, i = 1, \cdots, s$

i) $Re[\lambda - u \max_{1 \leqslant j \leqslant N} \frac{(a_j - 1)}{\alpha_j} - \sum_{i=1}^{r} \left[\sigma_i \frac{N_i}{M_i} \right] - \sum_{i=1}^{r} u_i \max_{1 \leqslant j \leqslant n_i} \frac{(c_j^{(i)} - 1)}{\gamma_j^{(i)}}] > 0$

ii) $Re[\mu - v \max_{1 \leqslant j \leqslant N} \frac{(a_j - 1)}{\alpha_j} - \sum_{i=1}^{r} \left[\eta_i \frac{N_i}{M_i} \right] - \sum_{i=1}^{r} v_i \max_{1 \leqslant j \leqslant n_i} \frac{(c_j^{(i)} - 1)}{\gamma_j^{(i)}}] > 0$

When $max(\sigma_i, \eta_i) < 0$, $i = 1, \cdots, s$

iii) $Re[\lambda - u \max_{1 \leqslant j \leqslant N} \frac{(a_j - 1)}{\alpha_j} - \sum_{i=1}^{r} u_i \max_{1 \leqslant j \leqslant n_i} \frac{(c_j^{(i)} - 1)}{\gamma_j^{(i)}}] > 0$

iV) $Re[\mu - v \max_{1 \leqslant j \leqslant N} \frac{(a_j - 1)}{\alpha_j} - \sum_{i=1}^{r} v_i \max_{1 \leqslant j \leqslant n_i} \frac{(c_j^{(i)} - 1)}{\gamma_j^{(i)}}] > 0$

When $\sigma_i \geqslant 0, \eta_i < 0, i = 1, \cdots, s$, the conditions i) and iV) are satisfied

When $\eta_i \geqslant 0, \sigma_i < 0, i = 1, \cdots, s$, the conditions ii) and iii) are satisfied

and the other conditions are given by d), e), f) and g) with the result (2.1)

Proof of (2.1)

To evaluate (2.1), first we use the serie representation for generalized hypergeometric function, general class of multivariable polynomials and Aleph-function of one variable given repectively by (1.7), (1.6) and (1.3) and express the generalized multivariable Aleph-function by the Mellin-Barnes contour type integral (1.9) in the left hand side of (2.1), then collect the power of $cx + d$ and $gx + f$. The binomial expansions for $x \in [a, b]$ is applied as follows :

$$(cx+d)^\gamma = (ac+d)^\gamma \sum_{l_1=0}^{\infty} \frac{(-\gamma)_{l_1}}{l_1!} \left\{ \frac{c(a-x)}{ac+d} \right\}^{l_1} \quad , \quad \left| \frac{(x-a)c}{ac+d} \right| < 1 \tag{2.8}$$

$$(gx+f)^\delta = (bg+f)^\delta \sum_{l_2=0}^{\infty} \frac{(-\delta)_{l_2}}{l_2!} \left\{ \frac{g(b-x)}{bg+f} \right\}^{l_2} \quad , \quad \left| \frac{(b-x)g}{bg+f} \right| < 1 \tag{2.9}$$

with γ and δ are replaced repectively by $\gamma - e's - p\eta_{G,g} - \sum_{i=1}^{s} \lambda_i K_i - \sum_{i=1}^{r} p_i s_i$ and

$\delta - h's - q\eta_{G,g} - \sum_{i=1}^{s} \mu_i K_i - \sum_{i=1}^{r} q_i s_i$. Now, we evaluate the innermost integral with the help of the following Eulerian beta type integral

$$\int_a^b (x-a)^{\lambda-1}(b-x)^{\mu-1}\mathrm{d}x = (b-a)^{\lambda+\mu-1}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}, Re(\lambda)>0, Re(\mu)>0, b\neq a \quad (2.10)$$

Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result. The proof of the integral (2.5) use the similar method.

3. Generalized multivariable I-function

If $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}} \rightarrow 1$, the generalized Aleph-function of several variables degenere to the generalized I-function of several variables and we have the following integrals.

First integral

$$\int_a^b (x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}{}_{p'}F_{q'}\Big((a_{p'});(b_{q'});w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\Big)$$

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}\Big(z\frac{(x-a)^u(b-x)^v}{(cx+d)^p(gx+f)^q}\Big)S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{y}_1\frac{(x-a)^{\sigma_1}(b-x)^{\eta_1}}{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}} \\ \cdots \\ \cdots \\ \mathrm{y}_s\frac{(x-a)^{\sigma_s}(b-x)^{\eta_s}}{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}} \end{array}\right) I^{m,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c} \mathrm{z}_1\frac{(x-a)^{u_1}(b-x)^{v_1}}{(cx+d)^{p_1}(gx+f)^{q_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_r\frac{(x-a)^{u_r}(b-x)^{v_r}}{(cx+d)^{p_r}(gx+f)^{q_r}} \end{array}\right)\mathrm{d}x$$

$$= (b-a)^{\lambda+\mu-1}(ac+d)^{\gamma}(bg+f)^{\delta}\sum_{l_1,l_2,s'=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{a'b'g'(\eta_{G,g})f(s')}{l_1!l_2!s'!}$$

$$\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}\Big(\frac{c(a-b)}{ac+d}\Big)^{l_1}\Big(\frac{g(b-a)}{bg+f}\Big)^{l_2} I^{m,\mathfrak{n}+4:V}_{U_{43}:W}\left(\begin{array}{c} \mathrm{z}_1\frac{(b-a)^{u_1+v_1}}{(ac+d)^{p_1}(bg+f)^{q_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_r\frac{(b-a)^{u_r+v_r}}{(ac+d)^{p_r}(bg+f)^{q_r}} \end{array}\right|$$

$$\begin{array}{l} (1\text{-}\lambda-u\eta_{G,g}-es'-l_1-\sum_{i=1}^s\sigma_iK_i:u_1,\cdots,u_r), \\ \quad\cdots \\ \quad (1+\gamma-p\eta_{G,g}-e's'-\sum_{i=1}^s\lambda_iK_i:p_1,\cdots,p_r), \end{array}$$

$$\begin{array}{l} (1\text{-}\mu-v\eta_{G,g}-hs'-l_2-\sum_{i=1}^s\eta_iK_i:v_1,\cdots,v_r), \\ \quad\cdots \\ \quad (1+\delta-q\eta_{G,g}-h's'-\sum_{i=1}^s\mu_iK_i:q_1,\cdots,q_r), \end{array}$$

$$\begin{array}{c} (1+\gamma-p\eta_{G,g}-e's'-l_1-\sum_{i=1}^s\lambda_iK_i:p_1,\cdots,p_r), \\ \cdots \\ (1\text{-}\lambda-\mu-(u+v)\eta_{G,g}-(e+h)s'-l_1-l_2-\sum_{i=1}^s(\sigma_i+\eta_i)K_i:u_1+v_1,\cdots,u_r+v_r), \end{array}$$

$$\left.\begin{array}{c} (1+\delta-q\eta_{G,g}-h's'-l_2-\sum_{i=1}^s\mu_iK_i:q_1,\cdots,q_r), A:B \\ \cdots \\ \mathrm{B}:\mathrm{D} \end{array}\right) \quad (3.1)$$

with the same notations and validity conditions that (2.1)

Second integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}}{(cx+d)^{\gamma}(gx+f)^{\delta}} {}_{p'}F_{q'}\left((a_{p'});(b_{q'});w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\right)\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(z\frac{(cx+d)^p(gx+f)^q}{(x-a)^u(b-x)^v}\right)$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{y}_1\frac{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}}{(x-a)^{\sigma_1}(b-x)^{\eta_1}} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_s\frac{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}}{(x-a)^{\sigma_s}(b-x)^{\eta_s}} \end{array}\right) I^{m,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c} \mathrm{z}_1\frac{(cx+d)^{p_1}(gx+f)^{q_1}}{(x-a)^{u_1}(b-x)^{v_1}} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r\frac{(cx+d)^{p_r}(gx+f)^{q_r}}{(x-a)^{u_r}(b-x)^{v_r}} \end{array}\right)\mathrm{d}x$$

$$=\frac{(b-a)^{\lambda+\mu-1}}{(ac+d)^{\gamma}(bg+f)^{\delta}}\sum_{l_1,l_2,s'=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{a'b''g'(\eta_{G,g})f(s')(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{l_1!l_2!s'!\qquad B_G g!}$$

$$\left(\frac{c(a-b)}{ac+d}\right)^{l_1}\left(\frac{g(b-a)}{bg+f}\right)^{l_2} I^{m+4,\mathfrak{n}:V}_{U_{34}:W}\left(\begin{array}{c} \mathrm{z}_1\frac{(ac+d)^{p_1}(bg+f)^{q_1}}{(b-a)^{u_1+v_1}} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r\frac{(ac+d)^{p_r}(bg+f)^{q_r}}{(b-a)^{u_r+v_r}} \end{array}\right|$$

$$(\gamma-p\eta_{G,g}+e's'-\textstyle\sum_{i=1}^s\lambda_iK_i:p_1,\cdots,p_r),\qquad(\delta-q\eta_{G,g}+h's'-\sum_{i=1}^s\mu_iK_i:q_1,\cdots,q_r),$$
$$\cdot\cdot\cdot\qquad\qquad\cdot\cdot\cdot$$
$$(\gamma+l_1+e's'-p\eta_{G,g}-\textstyle\sum_{i=1}^s\lambda_iK_i:p_1,\cdots,p_r),(\delta+l_2-q\eta_{G,g}+h's'-\sum_{i=1}^s\mu_iK_i:q_1,\cdots,q_r),$$

$$(\lambda+\mu+l_1+l_2+(e+h)s'-(u+v)\eta_{G,g}-\textstyle\sum_{i=1}^s(\sigma_i+\eta_i)K_i;u_1+v_1,\cdots,u_r+v_r),$$
$$\cdot\cdot\cdot$$
$$(\lambda+l_1+es'-u\eta_{G,g}-\textstyle\sum_{i=1}^s\sigma_iK_i:u_1,\cdots,u_r),$$

$$\left.\begin{array}{c}\mathrm{A}:\mathrm{C}\\ \cdot\cdot\cdot\\ (\mu+l_2+hs'-v\eta_{G,g}-\sum_{i=1}^s\eta_iK_i:v_1,\cdots,v_r),C:D\end{array}\right)\qquad\textbf{(3.2)}$$

with the same notations and validity conditions that (2.5)

4. Multivariable H-function

If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$and $R=R^{(1)}=,\cdots,R^{(r)}=1$the generalized Aleph-function of several variables degenere to the generalized H-function of several variables. And we have the following results

First integral

$$\int_a^b(x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}{}_{p'}F_{q'}\left((a_{p'});(b_{q'});w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\frac{(x-a)^u(b-x)^v}{(cx+d)^p(gx+f)^q}\right)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1\frac{(x-a)^{\sigma_1}(b-x)^{\eta_1}}{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}}\\ \cdots\\ \cdots\\ \mathrm{y}_s\frac{(x-a)^{\sigma_s}(b-x)^{\eta_s}}{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}}\end{array}\right)H_{p,q:W}^{m,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1\frac{(x-a)^{u_1}(b-x)^{v_1}}{(cx+d)^{p_1}(gx+f)^{q_1}}\\ \cdots\\ \cdots\\ \mathrm{z}_r\frac{(x-a)^{u_r}(b-x)^{v_r}}{(cx+d)^{p_r}(gx+f)^{q_r}}\end{array}\right)\mathrm{d}x$$

$$=(b-a)^{\lambda+\mu-1}(ac+d)^{\gamma}(bg+f)^{\delta}\sum_{l_1,l_2,s'=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{a'b'g'(\eta_{G,g})f(s')}{l_1!l_2!s'!}$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\left(\frac{c(a-b)}{ac+d}\right)^{l_1}\left(\frac{g(b-a)}{bg+f}\right)^{l_2}H_{p+4,q+3:W}^{m,\mathfrak{n}+4:V}\left(\begin{array}{c}\mathrm{z}_1\frac{(b-a)^{u_1+v_1}}{(ac+d)^{p_1}(bg+f)^{q_1}}\\ \cdots\\ \cdots\\ \mathrm{z}_r\frac{(b-a)^{u_r+v_r}}{(ac+d)^{p_r}(bg+f)^{q_r}}\end{array}\right|$$

$$(1\text{-}\lambda-u\eta_{G,g}-es'-l_1-\textstyle\sum_{i=1}^s\sigma_iK_i:u_1,\cdots,u_r),$$
$$\cdots$$
$$(1\text{+}\gamma-p\eta_{G,g}-e's'-\textstyle\sum_{i=1}^s\lambda_iK_i:p_1,\cdots,p_r),$$

$$(1\text{-}\mu-v\eta_{G,g}-hs'-l_2-\textstyle\sum_{i=1}^s\eta_iK_i:v_1,\cdots,v_r),$$
$$\cdots$$
$$(1\text{+}\delta-q\eta_{G,g}-h's'-\textstyle\sum_{i=1}^s\mu_iK_i:q_1,\cdots,q_r),$$

$$(1\text{+}\gamma-p\eta_{G,g}-e's'-l_1-\textstyle\sum_{i=1}^s\lambda_iK_i:p_1,\cdots,p_r),$$
$$\cdots$$
$$(1\text{-}\lambda-\mu-(u+v)\eta_{G,g}-(e+h)s'-l_1-l_2-\textstyle\sum_{i=1}^s(\sigma_i+\eta_i)K_i:u_1+v_1,\cdots,u_r+v_r),$$

$$\left.\begin{array}{c}(1\text{+}\delta-q\eta_{G,g}-h's'-l_2-\sum_{i=1}^s\mu_iK_i:q_1,\cdots,q_r),A:B\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{array}\right)\qquad\textbf{(4.1)}$$

with the same notations and validity conditions that (2.1)

Second integral

$$\int_a^b\frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}}{(cx+d)^{\gamma}(gx+f)^{\delta}}{}_{p'}F_{q'}\left((a_{p'});(b_{q'});w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\right)\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\frac{(cx+d)^p(gx+f)^q}{(x-a)^u(b-x)^v}\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1\frac{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}}{(x-a)^{\sigma_1}(b-x)^{\eta_1}}\\ \cdots\\ \cdots\\ \mathrm{y}_s\frac{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}}{(x-a)^{\sigma_s}(b-x)^{\eta_s}}\end{array}\right)H_{p,q:W}^{m,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1\frac{(cx+d)^{p_1}(gx+f)^{q_1}}{(x-a)^{u_1}(b-x)^{v_1}}\\ \cdots\\ \cdots\\ \mathrm{z}_r\frac{(cx+d)^{p_r}(gx+f)^{q_r}}{(x-a)^{u_r}(b-x)^{v_r}}\end{array}\right)\mathrm{d}x$$

$$= \frac{(b-a)^{\lambda+\mu-1}}{(ac+d)^{\gamma}(bg+f)^{\delta}} \sum_{l_1,l_2,s'=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{a'b''g'(\eta_{G,g})f(s')(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{l_1!l_2!s'!\,B_G g!}$$

$$\left(\frac{c(a-b)}{ac+d}\right)^{l_1}\left(\frac{g(b-a)}{bg+f}\right)^{l_2} H_{p+3,q+4:W}^{m+4,\mathfrak{n}:V}\left(\begin{array}{c|} \mathrm{z}_1 \frac{(ac+d)^{p_1}(bg+f)^{q_1}}{(b-a)^{u_1+v_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_r \frac{(ac+d)^{p_r}(bg+f)^{q_r}}{(b-a)^{u_r+v_r}} \end{array}\right.$$

$$(\gamma - p\eta_{G,g} + e's' - \textstyle\sum_{i=1}^{s} \lambda_i K_i : p_1, \cdots, p_r), \quad (\delta - q\eta_{G,g} + h's' - \textstyle\sum_{i=1}^{s} \mu_i K_i : q_1, \cdots, q_r),$$
$$\cdots \qquad \cdots$$
$$(\gamma + l_1 + e's' - p\eta_{G,g} - \textstyle\sum_{i=1}^{s} \lambda_i K_i : p_1, \cdots, p_r), (\delta + l_2 - q\eta_{G,g} + h's' - \textstyle\sum_{i=1}^{s} \mu_i K_i : q_1, \cdots, q_r),$$

$$(\lambda + \mu + l_1 + l_2 + (e+h)s' - (u+v)\eta_{G,g} - \textstyle\sum_{i=1}^{s} (\sigma_i + \eta_i) K_i ; u_1 + v_1, \cdots, u_r + v_r),$$
$$\cdots$$
$$(\lambda + l_1 + es' - u\eta_{G,g} - \textstyle\sum_{i=1}^{s} \sigma_i K_i : u_1, \cdots, u_r),$$

$$\left.\begin{array}{c} \mathrm{A} : \mathrm{C} \\ \cdots \\ (\mu + l_2 + hs' - v\eta_{G,g} - \sum_{i=1}^{s} \eta_i K_i : v_1, \cdots, v_r), C : D \end{array}\right) \tag{4.2}$$

with the same notations and validity conditions that (2.5)

5. Generalized Aleph-function of two variables

If $r = 2$, the generalized Aleph-function of several variables degenere to generalized Aleph-function of two variables . We have the following formulas

First integral

$$\int_a^b (x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}{}_{p'}F_{q'}\left((a_{p'});(b_{q'}); w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\frac{(x-a)^u(b-x)^v}{(cx+d)^p(gx+f)^q}\right) S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1 \frac{(x-a)^{\sigma_1}(b-x)^{\eta_1}}{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}} \\ \cdots \\ \cdots \\ \mathrm{y}_s \frac{(x-a)^{\sigma_s}(b-x)^{\eta_s}}{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}} \end{array}\right) \aleph_{U:W}^{m,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1 \frac{(x-a)^{u_1}(b-x)^{v_1}}{(cx+d)^{p_1}(gx+f)^{q_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_2 \frac{(x-a)^{u_2}(b-x)^{v_2}}{(cx+d)^{p_2}(gx+f)^{q_2}} \end{array}\right) \mathrm{d}x$$

$$= (b-a)^{\lambda+\mu-1}(ac+d)^{\gamma}(bg+f)^{\delta} \sum_{l_1,l_2,s'=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{a'b'g'(\eta_{G,g})f(s')}{l_1!l_2!s'!}$$

$$\frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\left(\frac{c(a-b)}{ac+d}\right)^{l_1}\left(\frac{g(b-a)}{bg+f}\right)^{l_2} \aleph_{U_{43}:W}^{m,\mathfrak{n}+4:V}\left(\begin{array}{c|} \mathrm{z}_1 \frac{(b-a)^{u_1+v_1}}{(ac+d)^{p_1}(bg+f)^{q_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_2 \frac{(b-a)^{u_2+v_2}}{(ac+d)^{p_2}(bg+f)^{q_2}} \end{array}\right.$$

$$(1\text{-}\lambda - u\eta_{G,g} - es' - l_1 - \sum_{i=1}^s \sigma_i K_i : u_1, u_2),(1\text{-}\mu - v\eta_{G,g} - hs' - l_2 - \sum_{i=1}^s \eta_i K_i : v_1, v_2),$$
$$\cdots \quad \cdots$$
$$(1+\gamma - p\eta_{G,g} - e's' - \sum_{i=1}^s \lambda_i K_i : p_1, p_2), \quad (1+\delta - q\eta_{G,g} - h's' - \sum_{i=1}^s \mu_i K_i : q_1, q_2),$$

$$(1+\gamma - p\eta_{G,g} - e's' - l_1 - \sum_{i=1}^s \lambda_i K_i : p_1, p_2),$$
$$\cdots$$
$$(1\text{-}\lambda - \mu - (u+v)\eta_{G,g} - (e+h)s' - l_1 - l_2 - \sum_{i=1}^s (\sigma_i + \eta_i) K_i : u_1 + v_1, u_2 + v_2),$$

$$\left.\begin{matrix}(1+\delta - q\eta_{G,g} - h's' - l_2 - \sum_{i=1}^s \mu_i K_i : q_1, q_2), A : B \\ \cdots \\ \mathrm{B} : \mathrm{D}\end{matrix}\right) \tag{5.1}$$

with the same notations and validity conditions that (2.1)

Second integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}}{(cx+d)^\gamma (gx+f)^\delta} {}_{p'}F_{q'}\left((a_{p'});(b_{q'}); w\frac{(x-a)^e(b-x)^h}{(cx+d)^{e'}(gx+f)^{h'}}\right) \aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\frac{(cx+d)^p(gx+f)^q}{(x-a)^u(b-x)^v}\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{matrix} \mathrm{y}_1 \frac{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}}{(x-a)^{\sigma_1}(b-x)^{\eta_1}} \\ \cdots \\ \cdots \\ \mathrm{y}_s \frac{(cx+d)^{\lambda_s}(gx+f)^{\mu_s}}{(x-a)^{\sigma_s}(b-x)^{\eta_s}} \end{matrix}\right) \aleph_{U:W}^{m,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \frac{(cx+d)^{p_1}(gx+f)^{q_1}}{(x-a)^{u_1}(b-x)^{v_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_2 \frac{(cx+d)^{p_2}(gx+f)^{q_2}}{(x-a)^{u_2}(b-x)^{v_2}} \end{matrix}\right) \mathrm{d}x$$

$$= \frac{(b-a)^{\lambda+\mu-1}}{(ac+d)^\gamma (bg+f)^\delta} \sum_{l_1,l_2,s'=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{a'b''g'(\eta_{G,g})f(s')(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{l_1! l_2! s'! \qquad B_G g!}$$

$$\left(\frac{c(a-b)}{ac+d}\right)^{l_1}\left(\frac{g(b-a)}{bg+f}\right)^{l_2} \aleph_{U_{34}:W}^{m+4,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 \frac{(ac+d)^{p_1}(bg+f)^{q_1}}{(b-a)^{u_1+v_1}} \\ \cdots \\ \cdots \\ \mathrm{z}_2 \frac{(ac+d)^{p_2}(bg+f)^{q_2}}{(b-a)^{u_2+v_2}} \end{matrix}\right|$$

$$(\gamma - p\eta_{G,g} + e's' - \sum_{i=1}^s \lambda_i K_i : p_1, p_2), \quad (\delta - q\eta_{G,g} + h's' - \sum_{i=1}^s \mu_i K_i : q_1, q_2),$$
$$\cdots \quad \cdots$$
$$(\gamma + l_1 + e's' - p\eta_{G,g} - \sum_{i=1}^s \lambda_i K_i : p_1, p_2),(\delta + l_2 - q\eta_{G,g} + h's' - \sum_{i=1}^s \mu_i K_i : q_1, q_2),$$

$$(\lambda + \mu + l_1 + l_2 + (e+h)s' - (u+v)\eta_{G,g} - \sum_{i=1}^s (\sigma_i + \eta_i) K_i; u_1 + v_1, u_2 + v_2),$$
$$\cdots$$
$$(\lambda + l_1 + es' - u\eta_{G,g} - \sum_{i=1}^s \sigma_i K_i : u_1, u_2),$$

$$\left.\begin{matrix} \mathrm{A} : \mathrm{C} \\ \cdots \\ (\mu + l_2 + hs' - v\eta_{G,g} - \sum_{i=1}^s \eta_i K_i : v_1, v_2), C : D \end{matrix}\right) \tag{5.2}$$

with the same notations and validity conditions that (2.5)

Remark : If $m = 0$ (see the first integral) then we obtain the multivariable I-function defined by Sharma et al [3], the multivariable H-function defined by Srivastava et al [6] and the Aleph-function of two variables defined by Sharma K [4]

7. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function ,multivariable Fox's H-function, Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

REFERENCES

[1] Ayant F.Y. An integral associated with the Aleph-functions of several variables. *International Journal of Mathematics Trends and Technology (IJMTT)*. 2016 Vol 31 (3), page 142-154.

[2] Chaurasia V.B.L and Singh Y. New generalization of integral equations of fredholm type using Aleph-function Int. J. of Modern Math. Sci. 9(3), 2014, p 208-220.

[3] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[4] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page1-13.

[5] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.

[6] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

[7] Südland N.; Baumann, B. and Nonnenmacher T.F. , Open problem : who knows about the Aleph-functions? Fract. Calc. Appl. Anal., 1(4) (1998): 401-402.

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