Eulerian integrals involving the multivariable Aleph-function I

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ABSTRACT
In this paper, we derive a general Eulerian integral involving the multivariable Aleph-function, Aleph-function of one variable and general class of polynomials of several variables. Some of this key formula could provide useful generalizations of some known as well as of some new results concerning the multivariable I-function and Aleph-function of two variables.

Keywords :multivariable Aleph-function, Eulerian integral, Multivariable I-function, Aleph-function of two variables, general class of polynomial,Aleph-function

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1. Introduction and preliminaries.

In this paper we establish a general Eulerian integral concerning the multivariable Aleph-function, the Aleph-function and general class of multivariable polynomials. The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [4] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $$\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]: \\ \cdot & \\ \cdot & \cdots\cdots\cdots,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]: \\ \cdot & \\ \mathrm{z}_r & \end{array}\right.$$

$$\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

For more details, see Ayant [2].
The reals numbers τ_i are positives for $i=1,\cdots,R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1,\cdots,R^{(k)}$
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with}\ k=1\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)} \tag{1.2}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1}\ldots|z_r|^{\alpha_r}\,)\,,max(\,|z_1|\ldots|z_r|\,)\rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1}\ldots|z_r|^{\beta_r}\,)\,,min(\,|z_1|\ldots|z_r|\,)\rightarrow \infty$$

where, with $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U=p_i,q_i,\tau_i;R\;;\;V=m_1,n_1;\cdots;m_r,n_r \tag{1.3}$$

$$\mathrm{W}=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.4}$$

$$A=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.5}$$

$$B=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.6}$$

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.7}$$

$$D=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.8}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r)=\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} z_1 & \mathrm{A}:\mathrm{C} \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ z_r & \mathrm{B}:\mathrm{D}\end{array}\right) \tag{1.9}$$

The generalized polynomials of multivariable defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s]=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.10}$$

The Aleph- function , introduced by Südland [8] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(z\;\middle|\;\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right)=\frac{1}{2\pi\omega}\int_L\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)z^{-s}\mathrm{d}s \tag{1.11}$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.12}$$

With $|argz|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0\,,i=1,\cdots,r$

For convergence conditions and other details of Aleph-function , see Südland et al [8].The serie representation of Aleph-function is given by Chaurasia et al [3].

$$\aleph_{P_i,Q_i,c_i;r}^{M,N}(z) = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_G g!}z^{-s} \tag{1.13}$$

With $s = \eta_{G,g} = \dfrac{b_G+g}{B_G}, P_i < Q_i, |z| < 1$ and $\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)$ is given in (1.2) (1.14)

2. Required formulas

We have : $B(\alpha,\beta) = \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ Re(\alpha) > 0, Re(\beta) > 0$ (2.1)

(2.1) can be rewritten in the form

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, Re(\alpha) > 0, Re(\beta) > 0, b \neq a \tag{2.2}$$

The binomial expansions for $t \in [a,b]$ yields :

$$(ut+v)^{\gamma} = (au+v)^{\gamma}\sum_{m=0}^{\infty}\frac{(-\gamma)_m}{m!}\left\{\frac{-u(a-t)}{au+v}\right\}^m \quad \text{where} \quad \left|\frac{(t-a)u}{au+v}\right| < 1 \tag{2.3}$$

With the help of (2.2) we obtain (see Srivastava et al [7])

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(at+v)^{\gamma}{}_2F_1\left(\begin{matrix}\alpha,-\gamma\\ \alpha+\beta\end{matrix};-\frac{(b-a)u}{au+v}\right) \tag{2.4}$$

3. The general Eulerian integral

Let $g_1(t) = \dfrac{(t-a)^{\delta_1}(b-t)^{\eta_1}(ut+v)^{1-\delta_1-\eta_1}}{B(ut+v)+(A-B)(t-a)}$; $g_2(t) = \dfrac{(t-a)^{\delta_2}(b-t)^{\eta_2}(yt+z)^{1-\delta_2-\eta_2}}{D(yt+z)+(C-D)(t-a)}$

$a' = \dfrac{(-N_1)_{M_1K_1}}{K_1!}\cdots\dfrac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s]$ and $U_{76} = p_i+7, q_i+6, \tau_i; R$

We shall derive the following general Eulerian integral involving the multivariable Aleph-function.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1\,(g_1(t))^{c_1}\,(g_2(t))^{d_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{y}_s\,(g_1(t))^{c_s}\,(g_2(t))^{d_s}\end{array}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(x\,(g_1(t))^{c}\,(g_2(t))^{d}\right)\aleph_{U:W}^{0,\mathrm{n}:V}\left(\begin{array}{c}\mathrm{z}_1\,(g_1(t))^{u_1}\,(g_2(t))^{v_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{z}_r\,(g_1(t))^{u_r}\,(g_2(t))^{v_r}\end{array}\right)\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(B-A/B)^l\,(D-C/D)^m}{l!m!k_1!k_2!}$$

$$a'\ \frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}Y_1^{K_1}\cdots y_s^{K_s}X^{\eta_{G,g}}\left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2}$$

$$\aleph^{0,\mathfrak{n}+7:V}_{U_{76}:W}\left(\begin{array}{c|c} \mathrm{Z}_1 & (1\text{-l-c}\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r),(1\text{-m-d}\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r),\\ \cdot\cdot\cdot & \cdot\cdot\cdot \qquad\qquad \cdot\cdot\cdot \\ \cdot\cdot\cdot & (1\text{-c}\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r),\quad(1\text{-d}\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r),\\ \mathrm{Z}_r & \end{array}\right.$$

$$(1\text{-}\alpha-l-m-k_1-(c\delta_1+d\delta_2)\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\delta_1c_i+\delta_2d_i)K_i:\delta_1u_1+\delta_2v_1,\cdots,\delta_1u_r+\delta_2v_r),$$
$$\cdot\cdot\cdot$$
$$\mathrm{B}_1,$$

$$(1\text{-}\beta-k_2-(c\eta_1+d\eta_2)\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\eta_1c_i+\eta_2d_i)K_i:\eta_1u_1+\eta_2v_1,\cdots,\eta_1u_r+\eta_2v_r),$$
$$\cdot\cdot\cdot$$
$$(1+\gamma-l-(\delta_1+\eta_1)c\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i;(\delta_1+\eta_1)u_1,\cdots,(\delta_1+\eta_1)u_r),$$

$$(1+\rho-m-k_2-(\eta_2+\delta_2)d\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),$$
$$\cdot\cdot\cdot$$
$$(1+\rho-m-(\eta_2+\delta_2)d\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),$$

$$\left.\begin{array}{c}(1+\gamma-l-k_1-(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^{s}(\eta_1+\delta_1)c_iK_i:(\eta_1+\delta_1)u_1,\cdots,(\eta_1+\delta_1)u_r),A_1,A:C\\ \cdot\cdot\cdot\\ \mathrm{B}_2,B:D\end{array}\right)\quad(3.1)$$

Where

$$B_1=(1\text{-}\alpha-\beta-m-c(\delta_1+\eta_1)\eta_{G,g}-d(\delta_2+\eta_2)\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i-\sum_{i=1}^{s}(\delta_2+\eta_2)d_iK_i;$$
$$(\delta_1+\eta_1)u_1+(\delta_2+\eta_2)v_1,\cdots,(\delta_1+\eta_1)u_r+(\delta_2+\eta_2)v_r)$$

$$A_1=(1\text{-}\alpha-\beta-l-m-c(\delta_1+\eta_1)\eta_{G,g}-d(\delta_2+\eta_2)\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i-\sum_{i=1}^{s}(\delta_2+\eta_2)d_iK_i;$$
$$(\delta_1+\eta_1)u_1+(\delta_2+\eta_2)v_1,\cdots,(\delta_1+\eta_1)u_r+(\delta_2+\eta_2)v_r)$$

$$B_2=(1\text{-}\alpha-\beta-m-l-k_1-k_2-c(\delta_1+\eta_1)\eta_{G,g}-d(\delta_2+\eta_2)\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i-$$
$$-\textstyle\sum_{i=1}^{s}(\delta_2+\eta_2)d_iK_i;(\delta_1+\eta_1)u_1+(\delta_2+\eta_2)v_1,\cdots,(\delta_1+\eta_1)u_r+(\delta_2+\eta_2)v_r)$$

$$Z_i=\frac{z_i(b-a)^{(\delta_1+\eta_1)u_i+(\delta_2+\eta_2)v_i}}{B^{u_i}D^{v_i}(au+v)^{(\delta_1+\eta_1)u_i}(by+z)^{(\delta_2+\eta_2)v_i}},i=1,\cdots,r$$

$$Y_i=\frac{y_i(b-a)^{(\delta_1+\eta_1)c_i+(\delta_2+\eta_2)d_i}}{B^{c_i}D^{d_i}(au+v)^{(\delta_1+\eta_1)c_i}(by+z)^{(\delta_2+\eta_2)d_i}},i=1,\cdots,s \text{ and}$$

$$X = \frac{x(b-a)^{(\delta_1+\eta_1)c+(\delta_2+\eta_2)d}}{B^c D^d (au+v)^{(\delta_1+\eta_1)c}(by+z)^{(\delta_2+\eta_2)d}}$$

Provided that

a) $min\{c,d,c_i,d_i,u_j,v_j\} > 0, i = 1,\cdots,s; j = 1,\cdots,r$

b $min\{Re(\alpha),Re(\beta)\} > 0, b \neq a$

c) $max\left\{\left|\frac{u(b-a)}{au+v}\right|,\left|\frac{y(b-a)}{by+z}\right|,\left|\frac{(t-a)(B-A)}{B(ut+v)}\right|,\left|\frac{(t-a)(D-C)}{D(yt+z)}\right|\right\} < 1, t \in [a,b]$

d) $Re\big[\alpha + (c\delta_1 + d\delta_2)\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{r}(\delta_1 u_i + \delta_2 v_i)\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\big] > -1$

e) $Re\big[\beta + (c\eta_1 + d\eta_2)\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{r}(\eta_1 u_i + \eta_2 v_i)\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\big] > -1$

f) $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.2)

g) $|argx| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

Proof of (3.1) Let $M = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$

We first replace the multivariable Aleph-function on the L.H.S of (3.1) by its Mellin-barnes contour integral (1.1), the Aleph-function and general class of polynomials of several variables in series using respectively (1.13) and (1.10), Now we interchange the order of summation and integrations (which is permissible under the conditions stated) . We get :

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a'\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}x^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}$$

$$(yt+z)^{\rho}\Big\{M\Big\{(g_1(t))^{c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i}(g_2(t))^{d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i}\Big\}\mathrm{d}s_1\cdots\mathrm{d}s_r\Big\}\mathrm{d}t \quad (3.2)$$

We evaluate the inner integrals with the help of (2.1) and (2.3) and applying (1.1)

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a'\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}x^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}$$

$$M\Big\{B^{-(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}\ D^{-(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\int_a^b (t-a)^{\alpha+\delta_1(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\delta_2(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(b-t)^{\beta+\eta_1(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\eta_2(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(ut+v)^{\gamma-(\delta_1+\eta_1)(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}(yt+z)^{\rho-(\delta_2+\eta_2)(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\left(1-\frac{(B-A)(t-a)}{B(ut+v)}\right)^{-(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}\left(1-\frac{(D-C)(t-a)}{D(yt+z)}\right)^{-(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\mathrm{d}t\Big\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.3}$$

Using binomial expansion (2.3) provided that $max\left\{\left|\frac{(t-a)(B-A)}{B(ut+v)}\right|\left|\frac{(t-a)(D-C)}{D(yt+z)}\right|\right\}<1, t\in[a,b]$

and also that the order of binomial summation and integration can be inversed, we get

$$\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}a'\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})(B-A/B)^l\,(D-C/D)^m}{B_Gg!\qquad\qquad l!m!}x^{\eta_{G,g}}$$

$$y_1^{K_1}\cdots y_s^{K_s}M\Big\{B^{-(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}D^{-(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\frac{\Gamma(l+c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)\Gamma(m+d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}{\Gamma(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)\Gamma(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\int_a^b(t-a)^{\alpha+l+m+\delta_1(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\delta_2(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(b-t)^{\beta+\eta_1(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\eta_2(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(ut+v)^{\gamma-l-(\delta_1+\eta_1)(c\eta_{G,g}+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}(yt+z)^{\rho-m-(\delta_2+\eta_2)(d\eta_{G,g}+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\mathrm{d}t\Big\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

The inner integral in (3.4) can be evaluated by using the following extension of Eulerian integral of Beta function given by Hussain and Srivastava [7].

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho}\mathrm{d}t=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)$$

$$\times F_3\left[\alpha,\beta,-\gamma,-\rho;\alpha+\beta;-\frac{(b-a)u}{au+v},\frac{(b-a)y}{by+z}\right] \tag{3.5}$$

where for convergence $min\{Re(\alpha),Re(\beta)\}>0, b\neq a$ and $max\left\{\left|\frac{u(b-a)}{au+v}\right|,\left|\frac{y(b-a)}{by+t}\right|\right\}<1$

and where F_3 denote the Appell function of two variables, see Appell et al [1]. Finally interpreting the resulting Mellin-Barnes contour integral as a multivariable Aleph-function, we obtain the desired result (3.1).

4. Multivariable I-function

If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\to 1$, the Aleph-function of several variables degenere to the I-function of several variables.

The general Eulerian integral have been derived in this section for multivariable I-functions defined by Sharma et al [4].

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho} S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{y}_1\,(g_1(t))^{c_1}\,(g_2(t))^{d_1} \\ \cdots \\ \cdots \\ \mathrm{y}_s\,(g_1(t))^{c_s}\,(g_2(t))^{d_s} \end{array}\right)$$

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(x\,(g_1(t))^{c}\,(g_2(t))^{d}\right) I^{0,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c} \mathrm{z}_1\,(g_1(t))^{u_1}\,(g_2(t))^{v_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r\,(g_1(t))^{u_r}\,(g_2(t))^{v_r} \end{array}\right)\mathrm{d}t = (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}$$

$$\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(B-A/B)^l\,(D-C/D)^m}{l!m!k_1!k_2!}\,a'\,\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}$$

$$Y_1^{K_1}\cdots Y_s^{K_s}X^{\eta_{G,g}}\left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2}$$

$$I^{0,\mathfrak{n}+7:V}_{U_{76}:W}\left(\begin{array}{c} \mathrm{Z}_1 \\ \cdots \\ \cdots \\ \mathrm{Z}_r \end{array}\left|\begin{array}{c} (\text{1-l-c}\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r),(\text{1-m-d}\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r), \\ \cdots \qquad \cdots \\ (\text{1-c}\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r),\quad(\text{1-d}\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r), \end{array}\right.\right.$$

$$\begin{array}{c}(1\text{-}\alpha-l-m-k_1-(c\delta_1+d\delta_2)\eta_{G,g}-\sum_{i=1}^{s}(\delta_1c_i+\delta_2d_i)K_i:\delta_1u_1+\delta_2v_1,\cdots,\delta_1u_r+\delta_2v_r), \\ \cdots \\ \mathrm{B}_1,\end{array}$$

$$\begin{array}{c}(1\text{-}\beta-k_2-(c\eta_1+d\eta_2)\eta_{G,g}-\sum_{i=1}^{s}(\eta_1c_i+\eta_2d_i)K_i:\eta_1u_1+\eta_2v_1,\cdots,\eta_1u_r+\eta_2v_r), \\ \cdots \\ (1+\gamma-l-(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i;(\delta_1+\eta_1)u_1,\cdots,(\delta_1+\eta_1)u_r),\end{array}$$

$$\begin{array}{c}(1+\rho-m-k_2-(\eta_2+\delta_2)d\eta_{G,g}-\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r), \\ \cdots \\ (1+\rho-m-(\eta_2+\delta_2)d\eta_{G,g}-\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),\end{array}$$

$$\left.\begin{array}{c}(1+\gamma-l-k_1-(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^{s}(\eta_1+\delta_1)c_iK_i:(\eta_1+\delta_1)u_1,\cdots,(\eta_1+\delta_1)u_r),A_1,A:C \\ \cdots \\ \mathrm{B}_2,B:D\end{array}\right)\qquad(4.1)$$

where the same notations and validity conditions that (3.1).

5. Aleph-function of two variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [5], and we have the following integral.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1 \left(g_1(t)\right)^{c_1} \left(g_2(t)\right)^{d_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_s \left(g_1(t)\right)^{c_s} \left(g_2(t)\right)^{d_s} \end{array}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(x\left(g_1(t)\right)^c \left(g_2(t)\right)^d\right) \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1 \left(g_1(t)\right)^{u_1} \left(g_2(t)\right)^{v_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_2 \left(g_1(t)\right)^{u_2} \left(g_2(t)\right)^{v_2} \end{array}\right) = (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}$$

$$\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} \frac{(B-A/B)^l (D-C/D)^m}{l!m!k_1!k_2!} a' \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}$$

$$Y_1^{K_1}\cdots Y_s^{K_s} X^{\eta_{G,g}} \left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2}$$

$$\aleph_{U_{76}:W}^{0,\mathfrak{n}+7:V}\left(\begin{array}{c} \mathrm{Z}_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{Z}_2 \end{array}\left|\begin{array}{c} (\text{1-l-c}\eta_{G,g}-\sum_{i=1}^s c_iK_i:u_1,u_2),(\text{1-m-d}\eta_{G,g}-\sum_{i=1}^s d_iK_i:v_1,v_2), \\ \cdot\cdot\cdot \qquad\qquad \cdot\cdot\cdot \\ (\text{1-c}\eta_{G,g}-\sum_{i=1}^s c_iK_i:u_1,u_2), \quad (\text{1-d}\eta_{G,g}-\sum_{i=1}^s d_iK_i:v_1,v_2), \end{array}\right.\right.$$

$$\begin{array}{c} (1\text{-}\alpha-l-m-k_1-(c\delta_1+d\delta_2)\eta_{G,g}-\sum_{i=1}^s(\delta_1c_i+\delta_2d_i)K_i:\delta_1u_1+\delta_2v_1,\delta_1u_2+\delta_2v_2), \\ \cdot\cdot\cdot \\ \mathrm{B}_1 \end{array}$$

$$\begin{array}{c} (1\text{-}\beta-k_2-(c\eta_1+d\eta_2)\eta_{G,g}-\sum_{i=1}^s(\eta_1c_i+\eta_2d_i)K_i:\eta_1u_1+\eta_2v_1,\eta_1u_2+\eta_2v_2), \\ \cdot\cdot\cdot \\ (1+\gamma-l-(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^s(\delta_1+\eta_1)c_iK_i;(\delta_1+\eta_1)u_1,(\delta_1+\eta_1)u_2) \end{array}$$

$$\begin{array}{c} (1+\rho-m-k_2-(\eta_2+\delta_2)d\eta_{G,g}-\sum_{i=1}^s(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,(\eta_2+\delta_2)v_2), \\ \cdot\cdot\cdot \\ (1+\rho-m-(\eta_2+\delta_2)d\eta_{G,g}-\sum_{i=1}^s(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,(\eta_2+\delta_2)v_2), \end{array}$$

$$\left.\begin{array}{c} (1+\gamma-l-k_1-(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^s(\eta_1+\delta_1)c_iK_i:(\eta_1+\delta_1)u_1,(\eta_1+\delta_1)u_2),A_1,A:C \\ \cdot\cdot\cdot \\ \mathrm{B}_2,B:D \end{array}\right) \quad (5.1)$$

where the same notations and validity conditions that (3.1) with $r=2$

Remark : If $\tau_i,\tau_i',\tau_i''\rightarrow 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [4] and we obtain the same formulae with the I-function of two variables

6. Conclusion

Due to general nature of the multivariable aleph-function and the Eulerian integral involving here, our formulas are capable to be reduced into many known and news integrals involving the special functions of one and several variables and polynomials of one and several variables.

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