Eulerian integral involving the multivariable Aleph-function III

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ABSTRACT
In this paper, we derive two Eulerian integrals involving a product of two multivariable Aleph-functions, Aleph-function of one variable and general class of polynomials of several variables. This general Eulerian integral formula is show to provide the key formula from which numerous others results for the multivariable Aleph-function, I-function of several variables and Aleph-function of two variables.

Keywords :multivariable Aleph-function, Eulerian integral, Multivariable I-function, Aleph-function of two variables, general class of polynomial,Aleph-function

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1. Introduction and preliminaries.

In this paper we establish two general Eulerian integrals concerning a product of two multivariable Aleph-functions, the Aleph-function and general class of multivariable polynomials. The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [3] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right.$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\cdots\cdots\cdots\cdots,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

For more details, see Ayant [1].
The reals numbers τ_i are positives for $i=1,\cdots,R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1,\cdots,R^{(k)}$
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with}\ k=1\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)} \tag{1.2}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1}\ldots|z_r|^{\alpha_r}\,)\,, max(\,|z_1|\ldots|z_r|\,) \rightarrow 0$$

$$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1}\ldots|z_r|^{\beta_r}\,)\,, min(\,|z_1|\ldots|z_r|\,) \rightarrow \infty$$

where, with $k = 1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j = 1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R\ ;\ V = m_1, n_1; \cdots; m_r, n_r \tag{1.3}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.4}$$

$$A = \{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.5}$$

$$B = \{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.6}$$

$$C = \{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.7}$$

$$D = \{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.8}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.9}$$

The generalized polynomials of multivariable defined by Srivastava [5], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s] y_1^{K_1} \cdots y_s^{K_s} \tag{1.10}$$

The Aleph- function , introduced by Südland [7] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\mathrm{z} \left| \begin{array}{c} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{array}\right.\right) = \frac{1}{2\pi\omega}\int_L \Omega_{P_i,Q_i,c_i;r}^{M,N}(s) z^{-s} \mathrm{d}s \tag{1.11}$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r}^{M,N}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r} c_i \prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.12}$$

With :

$$|argz| < \frac{1}{2}\pi\Omega \quad \text{Where } \Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0 \quad \text{with } i = 1,\cdots,r$$

For convergence conditions and other details of Aleph-function , see Südland et al [7]. Serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph_{P_i,Q_i,c_i;r}^{M,N}(z) = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_G g!}z^{-s} \tag{1.13}$$

$$\text{With } s = \eta_{G,g} = \frac{b_G + g}{B_G}, P_i < Q_i, |z| < 1 \text{ and } \Omega_{P_i,Q_i,c_i;r}^{M,N}(s) \text{ is given in (1.2)} \tag{1.14}$$

2. Required formulas

$$\text{We have : } B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad Re(\alpha) > 0, Re(\beta) > 0 \tag{2.1}$$

(2.1) can be rewritten in the form

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, Re(\alpha) > 0, Re(\beta) > 0, b \neq a \tag{2.2}$$

The binomial expansions for $t \in [a,b]$ yields :

$$(ut+v)^{\gamma} = (au+v)^{\gamma}\sum_{m=0}^{\infty}\frac{(-\gamma)_m}{m!}\left\{\frac{-u(a-t)}{au+v}\right\}^m \quad \text{where} \quad \left|\frac{(t-a)u}{au+v}\right| < 1 \tag{2.3}$$

With the help of (2.2) we obtain (see Srivastava et al [6])

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(au+v)^{\gamma}{}_2F_1\left(\begin{matrix}\alpha,-\gamma\\ \alpha+\beta\end{matrix};-\frac{(b-a)u}{au+v}\right) \tag{2.4}$$

$$\text{where } Re(\alpha) > 0, Re(\beta) > 0; \left|arg\left(\frac{bu+v}{au+v}\right)\right| \leqslant \pi - \epsilon(0 < \epsilon < \pi), b \neq a$$

3. General Eulerian integral of the multivariable Aleph-function

In this section, we shall prove two main general Eulerian integrals involving the Aleph-function of one variable, general class of polynomials and several variables and multivariable Aleph-function.

$$\text{We note : } a' = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s], \quad U_{11} = p_i+1, q_i+1, \tau_i; \mathfrak{R}$$

and $u_{11} = P_i+1, Q_i+1, \iota_i; \mathfrak{r}$. We have the following result :

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta}\ \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,\mathfrak{N}}\left(x(ut+v)^c(yt+z)^d\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{matrix}\mathrm{x}_1(ut+v)^{c_1}(yt+z)^{d_1}\\ \cdots\\ \cdots\\ \mathrm{x}_s(ut+v)^{c_s}(yt+z)^{d_s}\end{matrix}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{y}_1(ut+v)^{\sigma_1}\\ \cdots\\ \cdots\\ \mathrm{y}_r(ut+v)^{\sigma_r}\end{matrix}\right)\aleph_{u:w}^{0,N:v}\left(\begin{matrix}\mathrm{z}_1(yt+z)^{\lambda_1}\\ \cdots\\ \cdots\\ \mathrm{z}_R(yt+z)^{\lambda_R}\end{matrix}\right)\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}a'$$

$$\frac{(-)^g\Omega^{M,\mathfrak{N}}_{P_i,Q_i,\mathfrak{c}_i,r'}(\eta_{G,g})}{B_G g!}X_1^{K_1}\cdots X_s^{K_s}X^{\eta_{G,g}}\left\{\frac{(b-a)u}{(au+v)}\right\}^l\left\{-\frac{(b-a)y}{(by+z)}\right\}^m$$

$$\aleph^{0,\mathfrak{n}+1:V}_{U_{11}:W}\left(\begin{array}{c}\mathrm{y}_1(au+v)^{\sigma_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{y}_r(au+v)^{\sigma_r}\end{array}\middle|\begin{array}{c}(-\gamma-c\eta_{G,g}-\sum_{i=1}^s c_iK_i:\sigma_1,\cdots,\sigma_r),A:C\\ \cdot\cdot\cdot\\ (-\gamma+l-c\eta_{G,g}-\sum_{i=1}^s c_iK_i:\sigma_1,\cdots,\sigma_r),B:D\end{array}\right)$$

$$\aleph^{0,N+1:v}_{u_{11}:w}\left(\begin{array}{c}\mathrm{z}_1(by+z)^{\lambda_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{z}_R(by+z)^{\lambda_R}\end{array}\middle|\begin{array}{c}(-\delta-d\eta_{G,g}-\sum_{i=1}^s d_iK_i:\lambda_1,\cdots,\lambda_R),A':C'\\ \cdot\cdot\cdot\\ (-\delta+m-d\eta_{G,g}-\sum_{i=1}^s d_iK_i:\lambda_1,\cdots,\lambda_R),B':D'\end{array}\right) \quad (3.1)$$

where $X = x(au+v)^c\ (by+z)^d$ and $X_i = x_i(au+v)^{c_i}\ (by+z)^{d_i}, i=1,\cdots,s$

Provided that

a) $min\{c,d,c_i,d_i,\sigma_j,\lambda_k\} > 0, i=1,\cdots,s; j=1,\cdots,r; k=1,\cdots,R$

b $min\{Re(\alpha),Re(\beta)\} > 0; b \neq a$, $max\left\{\left|\frac{u(b-a)}{au+v}\right|,\left|\frac{y(b-a)}{by+z}\right|\right\} < 1$

c) $Re\left[\gamma + c\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^r \sigma_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > -1$

d) $Re\left[\delta + d\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^r \lambda_i \min_{1\leqslant j\leqslant m_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}\right] > -1$

e) $|argy_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.2) and e) $|argZ_k| < \frac{1}{2}B_i^{(k)}\pi$

f) $|argx| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^M \beta_j + \sum_{j=1}^N \alpha_j - \mathfrak{c}_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=\mathfrak{N}+1}^{P_i}\alpha_{ji}) > 0$

Proof

We first replace the two multivariable Aleph-functions on the L.H.S of (3.1) by its Mellin-barnes contour integral respectively, see Ayant [1], the Aleph-function and general class of polynomials of several variables in series using respectively (1.13) and (1.10),Now we interchange the order of summation and integrations (which is permissible under the conditions stated). Collect the powers of $(ut+v),(yt+z)$ and apply the binomial expansion (2.3). We then use the Eulerian integral (2.2) and interpret the resulting both Mellin-Barnes contour integral as an Aleph-function of r variables and an Aleph-function of R variables respectively, we arrive at the desired result.

In (3.1) replace a by $-a$ and y by $-y$, we obtain the following Eulerian integral :

$$\int_{-a}^{b}(t+a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\ (z-ty)^{\delta}\ \aleph^{M,\mathfrak{N}}_{P_i,Q_i,\mathfrak{c}_i;r'}\big(x(ut+v)^c(z-ty)^d\big)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{x}_1(ut+v)^{c_1}(z-ty)^{d_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{x}_s(ut+v)^{c_s}(z-ty)^{d_s} \end{pmatrix} \aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{y}_1(ut+v)^{\sigma_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_r(ut+v)^{\sigma_r} \end{pmatrix} \aleph_{u:w}^{0,N:v}\begin{pmatrix} \mathrm{z}_1(z-ty)^{\lambda_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_R(z-ty)^{\lambda_R} \end{pmatrix}\mathrm{d}t$$

$$=(b+a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}a'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,\mathfrak{c}_i,r'}^{M,\mathfrak{N}}(\eta_{G,g})}{B_G g!}X_1^{K_1}\cdots X_s^{K_s}X^{\eta_{G,g}}\left\{-\frac{(b+a)u}{(bu+v)}\right\}^l\left\{-\frac{(b+a)y}{(ay+z)}\right\}^m$$

$$\aleph_{U_{11}:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c} \mathrm{y}_1(bu+v)^{\sigma_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_r(bu+v)^{\sigma_r} \end{array}\middle| \begin{array}{c} (\text{-}\gamma-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\cdots,\sigma_r),A:C \\ \cdot\cdot\cdot \\ (\text{-}\gamma+l-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\cdots,\sigma_r),B:D \end{array}\right)$$

$$\aleph_{u_{11}:w}^{0,N+1:v}\left(\begin{array}{c} \mathrm{z}_1(ay+z)^{\lambda_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_R(ay+z)^{\lambda_R} \end{array}\middle| \begin{array}{c} (\text{-}\delta-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\cdots,\lambda_R),A':C' \\ \cdot\cdot\cdot \\ (\text{-}\delta+m-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\cdots,\lambda_R),B':D' \end{array}\right) \quad (3.2)$$

Provided that : $max\left\{\left|\frac{u(b+a)}{bu+v}\right|,\left|\frac{y(b+a)}{ay+z}\right|\right\}<1$. The other validity conditions are the sames that (3.1) and the notations are similar that (3.1)

4. Multivariable I-function

If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}},\iota,\iota_{i^{(1)}}\cdots,\iota_{i^{(R)}}\rightarrow 1$, the Aleph-function of several variables degenere to the I-function of several variables. The two general Eulerian integral have been derived in this section for multivariable I-functions defined by Sharma et al [3].

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\ (yt+z)^{\delta}\ \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,\mathfrak{N}}\left(x(ut+v)^c(yt+z)^d\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{x}_1(ut+v)^{c_1}(yt+z)^{d_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{x}_s(ut+v)^{c_s}(yt+z)^{d_s} \end{pmatrix} I_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{y}_1(ut+v)^{\sigma_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_r(ut+v)^{\sigma_r} \end{pmatrix} I_{u:w}^{0,N:v}\begin{pmatrix} \mathrm{z}_1(yt+z)^{\lambda_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_R(yt+z)^{\lambda_R} \end{pmatrix}\mathrm{d}t$$

$$=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}a'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,\mathfrak{c}_i,r'}^{M,\mathfrak{N}}(\eta_{G,g})}{B_G g!}X_1^{K_1}\cdots X_s^{K_s}X^{\eta_{G,g}}\left\{\frac{(b-a)u}{(au+v)}\right\}^l\left\{-\frac{(b-a)y}{(by+z)}\right\}^m$$

$$I_{U_{11}:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c} y_1(au+v)^{\sigma_1} \\ \cdots \\ \cdots \\ y_r(au+v)^{\sigma_r} \end{array}\middle| \begin{array}{c} (-\gamma-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\cdots,\sigma_r),A:C \\ \cdots \\ (-\gamma+l-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\cdots,\sigma_r),B:D \end{array}\right)$$

$$I_{u_{11}:w}^{0,N+1:v}\left(\begin{array}{c} z_1(by+z)^{\lambda_1} \\ \cdots \\ \cdots \\ z_R(by+z)^{\lambda_R} \end{array}\middle| \begin{array}{c} (-\delta-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\cdots,\lambda_R),A':C' \\ \cdots \\ (-\delta+m-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\cdots,\lambda_R),B':D' \end{array}\right) \quad (3.1)$$

where the same notations and validity conditions that (3.1).

$$\int_{-a}^{b}(t+a)^{\alpha-1}(b-t)^{\beta-1}\,(ut+v)^{\gamma}\,\,(z-ty)^{\delta}\,\,\aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,\mathfrak{N}}(x(ut+v)^{c}(z-ty)^{d})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} x_1(ut+v)^{c_1}(z-ty)^{d_1} \\ \cdots \\ \cdots \\ x_s(ut+v)^{c_s}(z-ty)^{d_s} \end{array}\right) I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} y_1(ut+v)^{\sigma_1} \\ \cdots \\ \cdots \\ y_r(ut+v)^{\sigma_r} \end{array}\right)\aleph_{u:w}^{0,N:v}\left(\begin{array}{c} z_1(z-ty)^{\lambda_1} \\ \cdots \\ \cdots \\ z_R(z-ty)^{\lambda_R} \end{array}\right)\mathrm{d}t$$

$$=(b+a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}a'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,\mathfrak{c}_i,r'}^{M,\mathfrak{N}}(\eta_{G,g})}{B_G g!}X_1^{K_1}\cdots X_s^{K_s}X^{\eta_{G,g}}\left\{-\frac{(b+a)u}{(bu+v)}\right\}^l\left\{-\frac{(b+a)y}{(ay+z)}\right\}^m$$

$$I_{U_{11}:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c} y_1(bu+v)^{\sigma_1} \\ \cdots \\ \cdots \\ y_r(bu+v)^{\sigma_r} \end{array}\middle| \begin{array}{c} (-\gamma-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\cdots,\sigma_r),A:C \\ \cdots \\ (-\gamma+l-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\cdots,\sigma_r),B:D \end{array}\right)$$

$$I_{u_{11}:w}^{0,N+1:v}\left(\begin{array}{c} z_1(ay+z)^{\lambda_1} \\ \cdots \\ \cdots \\ z_R(ay+z)^{\lambda_R} \end{array}\middle| \begin{array}{c} (-\delta-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\cdots,\lambda_R),A':C' \\ \cdots \\ (-\delta+m-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\cdots,\lambda_R),B':D' \end{array}\right) \quad (4.2)$$

Provided that : $max\left\{\left|\frac{u(b+a)}{bu+v}\right|,\left|\frac{y(b+a)}{ay+z}\right|\right\}<1.$ The other validity conditions are the sames that (4.1) and the notations are similar that (4.1).

5. Aleph-function of two variables

If $r=R=2$, we obtain two Aleph-functions of two variables defined by K.Sharma [4], and we have the following integrals.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\,(ut+v)^{\gamma}\;(yt+z)^{\delta}\;\aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,\mathfrak{N}}\big(x(ut+v)^{c}(yt+z)^{d}\big)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{x}_1(ut+v)^{c_1}(yt+z)^{d_1}\\ \cdots\\ \cdots\\ \mathrm{x}_s(ut+v)^{c_s}(yt+z)^{d_s}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{y}_1(ut+v)^{\sigma_1}\\ \cdots\\ \cdots\\ \mathrm{y}_2(ut+v)^{\sigma_2}\end{array}\right)\aleph_{u:w}^{0,N:v}\left(\begin{array}{c}\mathrm{z}_1(yt+z)^{\lambda_1}\\ \cdots\\ \cdots\\ \mathrm{z}_2(yt+z)^{\lambda_2}\end{array}\right)\mathrm{d}t$$

$$=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}a'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,\mathfrak{c}_i,r'}^{M,\mathfrak{N}}(\eta_{G,g})}{B_G g!}X_1^{K_1}\cdots X_s^{K_s}X^{\eta_{G,g}}\left\{\frac{(b-a)u}{(au+v)}\right\}^l\left\{-\frac{(b-a)y}{(by+z)}\right\}^m$$

$$\aleph_{U_{11}:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c}\mathrm{y}_1(au+v)^{\sigma_1}\\ \cdots\\ \cdots\\ \mathrm{y}_2(au+v)^{\sigma_2}\end{array}\middle|\begin{array}{c}(\text{-}\gamma-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\sigma_2),A:C\\ \cdots\\ (\text{-}\gamma+l-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\sigma_2),B:D\end{array}\right)$$

$$\aleph_{u_{11}:w}^{0,N+1:v}\left(\begin{array}{c}\mathrm{z}_1(by+z)^{\lambda_1}\\ \cdots\\ \cdots\\ \mathrm{z}_2(by+z)^{\lambda_2}\end{array}\middle|\begin{array}{c}(\text{-}\delta-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\lambda_2),A':C'\\ \cdots\\ (\text{-}\delta+m-d\eta_{G,g}-\sum_{i=1}^{s}d_iK_i:\lambda_1,\lambda_2),B':D'\end{array}\right)\tag{5.1}$$

where the same notations and validity conditions that (3.1) with $\mathfrak{R}=\mathfrak{r}=2$

$$\int_{-a}^b (t+a)^{\alpha-1}(b-t)^{\beta-1}\,(ut+v)^{\gamma}\;(z-ty)^{\delta}\;\aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,\mathfrak{N}}\big(x(ut+v)^{c}(z-ty)^{d}\big)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{x}_1(ut+v)^{c_1}(z-ty)^{d_1}\\ \cdots\\ \cdots\\ \mathrm{x}_s(ut+v)^{c_s}(z-ty)^{d_s}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{y}_1(ut+v)^{\sigma_1}\\ \cdots\\ \cdots\\ \mathrm{y}_2(ut+v)^{\sigma_2}\end{array}\right)\aleph_{u:w}^{0,N:v}\left(\begin{array}{c}\mathrm{z}_1(z-ty)^{\lambda_1}\\ \cdots\\ \cdots\\ \mathrm{z}_2(z-ty)^{\lambda_2}\end{array}\right)\mathrm{d}t$$

$$=(b+a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}a'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,\mathfrak{c}_i,r'}^{M,\mathfrak{N}}(\eta_{G,g})}{B_G g!}X_1^{K_1}\cdots X_s^{K_s}X^{\eta_{G,g}}\left\{-\frac{(b+a)u}{(bu+v)}\right\}^l\left\{-\frac{(b+a)y}{(ay+z)}\right\}^m$$

$$\aleph_{U_{11}:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c}\mathrm{y}_1(bu+v)^{\sigma_1}\\ \cdots\\ \cdots\\ \mathrm{y}_2(bu+v)^{\sigma_2}\end{array}\middle|\begin{array}{c}(\text{-}\gamma-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\sigma_2),A:C\\ \cdots\\ (\text{-}\gamma+l-c\eta_{G,g}-\sum_{i=1}^{s}c_iK_i:\sigma_1,\sigma_2),B:D\end{array}\right)$$

$$\aleph_{u_{11}:w}^{0,N+1:v}\left(\begin{array}{c} z_1(ay+z)^{\lambda_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ z_2(ay+z)^{\lambda_2} \end{array}\middle| \begin{array}{c} (\text{-}\delta - d\eta_{G,g} - \sum_{i=1}^{s} d_i K_i : \lambda_1, \lambda_2), A' : C' \\ \cdot\ \cdot\ \cdot \\ (\text{-}\delta + m - d\eta_{G,g} - \sum_{i=1}^{s} d_i K_i : \lambda_1, \lambda_2), B' : D' \end{array}\right) \tag{5.2}$$

Provided that : $max\left\{\left|\frac{u(b+a)}{bu+v}\right|, \left|\frac{y(b+a)}{ay+z}\right|\right\} < 1.$ The other validity conditions are the sames that (3.1) and the notations are similar that (3.1) with $\mathfrak{R} = \mathfrak{r} = 2$.

6. Conclusion

Due to general nature of the multivariable aleph-function and the Eulerian integral involving here, our formulas are capable to be reduced into many known and news integrals involving the special functions of one and several variables and polynomials of one and several variables.

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