

An expansion formula for the multivariable Aleph-function

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ABSTRACT

The aim of the paper is to establish a general expansion formula for the Aleph-function of several variables. It is significant to observe that a large number of finite and infinite series for this function can be easily summed up by using the summation theorem for ordinary hypergeometric series in the main result. Also, by appropriately specializing the parameters of the multivariable Aleph-function, one can obtain expansion formulas for simpler special functions of one and several variables.

Keywords: Multivariable Aleph-function, Mellin-Barnes contour, generalized hypergeometric function, expansion formula, I-function of several variables, Aleph-function of two variables.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction and preliminaries.

We begin by recalling the definition of the Aleph-function of several variables. The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [3], itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariable Aleph-function throughout our present study and will be defined and represented as follows.

$$\text{We have : } \aleph(z_1, \dots, z_r) = \aleph_{p_i, q_i, \tau_i; R; p_i^{(1)}, q_i^{(1)}, \tau_i^{(1)}; R^{(1)}; \dots; p_i^{(r)}, q_i^{(r)}, \tau_i^{(r)}; R^{(r)}}^{0, n; m_1, n_1, \dots, m_r, n_r} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \right)$$

$$[(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)})_{1, n}] , [\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(r)})_{n+1, p_i}] :$$

$$\dots, [\tau_i(b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(r)})_{m+1, q_i}] :$$

$$\left(\begin{matrix} [(c_j^{(1)}, \gamma_j^{(1)})_{1, n_1}], [\tau_{i(1)}(c_{ji}^{(1)}, \gamma_{ji}^{(1)})_{n_1+1, p_i^{(1)}}]; \dots ; [(c_j^{(r)}, \gamma_j^{(r)})_{1, n_r}], [\tau_{i(r)}(c_{ji}^{(r)}, \gamma_{ji}^{(r)})_{n_r+1, p_i^{(r)}}] \\ [(d_j^{(1)}, \delta_j^{(1)})_{1, m_1}], [\tau_{i(1)}(d_{ji}^{(1)}, \delta_{ji}^{(1)})_{m_1+1, q_i^{(1)}}]; \dots ; [(d_j^{(r)}, \delta_j^{(r)})_{1, m_r}], [\tau_{i(r)}(d_{ji}^{(r)}, \delta_{ji}^{(r)})_{m_r+1, q_i^{(r)}}] \end{matrix} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} ds_1 \dots ds_r \quad (1.1)$$

with $\omega = \sqrt{-1}$

For more details, see Ayant [2].

The reals numbers τ_i are positives for $i = 1, \dots, R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1, \dots, R^{(k)}$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^n \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \dots, r, i = 1, \dots, R, i^{(k)} = 1, \dots, R^{(k)} \quad (1.2)$$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\alpha_1} \dots |z_r|^{\alpha_r}), \max(|z_1| \dots |z_r|) \rightarrow 0$$

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\beta_1} \dots |z_r|^{\beta_r}), \min(|z_1| \dots |z_r|) \rightarrow \infty$$

where, with $k = 1, \dots, r : \alpha_k = \min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \dots, m_k$ and

$$\beta_k = \max[Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \dots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R; V = m_1, n_1; \dots; m_r, n_r \tag{1.3}$$

$$W = p_{i(1)}, q_{i(1)}, \tau_{i(1)}; R^{(1)}, \dots, p_{i(r)}, q_{i(r)}, \tau_{i(r)}; R^{(r)} \tag{1.4}$$

$$A = \{(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.5}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.6}$$

$$C_1 = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i(1)}(c_{ji(1)}; \gamma_{ji(1)})_{n_1+1,p_{i(1)}}, \dots,$$

$$C_r = \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i(r)}(c_{ji(r)}; \gamma_{ji(r)})_{n_r+1,p_{i(r)}} \tag{1.7}$$

$$D_1 = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i(1)}(d_{ji(1)}; \delta_{ji(1)})_{m_1+1,q_{i(1)}}, \dots,$$

$$D_r = \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i(r)}(d_{ji(r)}; \delta_{ji(r)})_{m_r+1,q_{i(r)}} \tag{1.8}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \dots, z_r) = \aleph_{U;W}^{0,n;V} \left(\begin{array}{c|c} z_1 & A; C_1; \dots; C_r \\ \cdot & \cdot \cdot \cdot \\ \cdot & \cdot \cdot \cdot \\ z_r & B; D_1; \dots; D_r \end{array} \right) \tag{1.9}$$

Let $W = W^{(1)}; \dots; W^{(r)}$ where $W^{(k)} = p_{i(k)}, q_{i(k)}, \tau_{i(k)}; R^{(k)}$. $C = C_1; \dots; C_r$ and $D = D_1; \dots; D_r$

We note for exempt: $W_{EF}^{(k)} = p_{i(k)} + E; q_{i(k)} + F, \tau_{i(k)}; R^{(k)}$ and $U_{C+D,E} = p_i + C + D, q_i + E, \tau_i; R$

2. General expansion formula

We have the following general formula

$$\sum_{k=1}^{\infty} \frac{\prod_{j=1}^{A'} (a'_j)_k z^k}{\prod_{j=1}^{B'} (b'_j)_k k!} \aleph_{U_{C+D,E}; W_{G_1+H_1, F_1+l_1}; \dots; W_{G_r+H_r, F_r+l_r}}^{0, n+C; m_1+F_1, n_1+G_1; \dots; m_r+F_r, n_r+G_r} \left(\begin{array}{c|c} z_1 & \\ \cdot & \\ \cdot & \\ z_r & \end{array} \right)$$

$$A, (u_j - k; \rho'_j, \dots, \rho_j^{(r)})_{1,C}, (v_j + k; \sigma'_j, \dots, \sigma_j^{(r)})_{1,D} : C_1, (g'_j - k; \eta'_j)_{1,G_1}, (h'_j + k; \xi'_j)_{1,H_1};$$

$$\dots \dots \dots$$

$$B, (w_j - k; \epsilon'_j, \dots, \epsilon_j^{(r)})_{1,E}, \dots \dots : D_1, (i'_j + k; \mu'_j)_{1,F_1}, (l'_j - k; \nu'_j)_{1,l_1};$$

$$\left(\begin{matrix} \dots ; C_r, (g_j^{(r)} - k; \eta_j^{(r)})_{1,G_r}, (h_j^{(r)} + k; \xi_j^{(r)})_{1,H_r} \\ \dots \\ \dots ; D_r, (i_j^{(r)} + k; \mu_j^{(r)})_{1,F_r}, (l_j^{(r)} - k; \nu_j^{(r)})_{1,l_r} \end{matrix} \right) = \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k'=1}^r \theta_{k'}(s_{k'})$$

$$\Gamma \left(\begin{matrix} A_1(s_1, \dots, s_r) \\ \dots \\ B_1(s_1, \dots, s_r) \end{matrix} \right)_{A'+PF_{B'+Q}} \left(\begin{matrix} (a_{A'}), A_1(s_1, \dots, s_r) \\ \dots \\ (b_{B'}), B_1(s_1, \dots, s_r) \end{matrix} ; z \right) z_1^{s_1} \dots z_r^{s_r} ds_1 \dots ds_r \quad (2.1)$$

where $A, B, C_1, \dots, C_r, D_1, \dots, D_r$ are defined by (1.5), (1.6), (1.7) and (1.8) respectively.

$$A_1(s_1, \dots, s_r) \text{ stands for } (1 - u_j + \sum_{i=1}^r \rho_j^{(i)} s_i)_{1,C}, (1 - g'_j + \eta'_j s_1)_{1,G_1}, \dots,$$

$$(1 - g_j^{(r)} + \eta_j^{(r)} s_r)_{1,G_r}, (i'_j - \mu'_j s_1)_{1,F_1}, \dots, (i_j^{(r)} - \mu_j^{(r)} s_r)_{1,F_r} \quad (2.2)$$

$$B_1(s_1, \dots, s_r) \text{ stands for } (v_j - \sum_{i=1}^r \sigma_j^{(i)} s_i)_{1,D}, (1 - w_j + \sum_{i=1}^r \epsilon_j^{(i)} s_i)_{1,E}, (h'_j - \xi'_j s_i)_{1,H_1},$$

$$(h_j^{(r)} - \xi_j^{(r)} s_r)_{1,H_r}, (1 - l'_j + \nu'_j s_1)_{1,l_1}, \dots, (1 - l_j^{(r)} + \nu_j^{(r)} s_r)_{1,l_r} \quad (2.3)$$

The formula (2.1) holds if any one of the following sets of conditions is satisfied.

- a) $A' + P < B' + Q + 1$
- b) $A' + P = B' + Q + 1, |z| < 1$
- c) $A' + P = B' + Q + 1, z = 1, Re(\theta) > 0$
- d) $A' + P = B' + Q + 1, z = -1, Re(\theta + 1) > 0$

$$\text{where } P = C + \sum_{i=1}^r (G_i + F_i) \text{ and } Q = D + E + \sum_{i=1}^r (H_i + l_i) \quad (2.4)$$

$$\text{and } \theta = \sum_{j=1}^{B'} b'_j - \sum_{j=1}^{A'} a'_j - \left\{ \sum_{i=1}^r \left(\sum_{j=1}^{F_i} i_j^{(r)} + \sum_{j=1}^{l_i} l_j^{(r)} - \sum_{j=1}^{G_i} g_j^{(r)} - \sum_{j=1}^{H_i} h_j^{(r)} \right) + \right.$$

$$\left. \sum_{j=1}^E w_j - \sum_{j=1}^C u_j - \sum_{j=1}^D v_j \right\} + E - C + \sum_{i=1}^r (l_i - G_i) \quad (2.5)$$

Proof of (2.1)

To evaluate (2.1), we first replace the multivariable Aleph-function occurring on the left-hand side of (2.1) by the Mellin-Barnes contour integral (1.1), and change the order of summation and integration (which is permissible under the conditions stated), then we get the desired result (2.1) on using the definition of the generalized hypergeometric function (see [2],page 40).

3. Particular cases

a) If in (2.1) $A' = 2, B' = P = 0, E = Q = 1, z = 1$, using the Gauss theorem ([6], page 243 (III.3)), we get the following summation formula :

$$\sum_{k=0}^{\infty} \frac{(a'_1)_k (a'_2)_k}{k!} \mathbb{N}_{U_{01}:W}^{0,n:V} \left(\begin{matrix} z_1 \\ \vdots \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} A : C \\ \vdots \\ (w_1 - k; \epsilon'_1, \dots, \epsilon_1^{(r)}), B : D \end{matrix} \right) = \mathbb{N}_{U_{12}:W}^{0,n+1:V} \left(\begin{matrix} z_1 \\ \vdots \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (w_1 + a'_1 + a'_2; \epsilon'_1, \dots, \epsilon_1^{(r)}), A : C \\ \vdots \\ (w_1 + a'_1; \epsilon'_1, \dots, \epsilon_1^{(r)}), (w_1 + a'_2; \epsilon'_1, \dots, \epsilon_1^{(r)}), B : D \end{matrix} \right) \tag{3.1}$$

where $U_{01} = p_i, q_i + 1, \tau_i; R$ and $U_{12} = p_i + 1, q_i + 2, \tau_i; R$

b) If in (2.1) $A' = 1, B' = 0, P = Q = 1 = E = C = 1, z = -1$, and use the Kummer's theorem ([6], page 243 (III.5)), we have the following result :

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mathbb{N}_{U_{11}:W}^{0,n+1:V} \left(\begin{matrix} z_1 \\ \vdots \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (u_1 - k; \rho'_1, \dots, \rho_1^{(r)}), A : C \\ \vdots \\ (-1 + a'_1 + u_1 - k; \rho'_1, \dots, \rho_1^{(r)}), B : D \end{matrix} \right) = \mathbb{N}_{U_{22}:W}^{0,n+2:V} \left(\begin{matrix} z_1 \\ \vdots \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (u_1; \rho'_1, \dots, \rho_1^{(r)}), (\frac{u_1-1}{2}; \frac{\rho'_1}{2}, \dots, \frac{\rho_1^{(r)}}{2}), A : C \\ \vdots \\ (-1 + u_1 - k; \rho'_1, \dots, \rho_1^{(r)}), (\frac{u_1-1}{2} + a'_1; \frac{\rho'_1}{2}, \dots, \frac{\rho_1^{(r)}}{2}), B : D \end{matrix} \right) \tag{3.2}$$

where $U_{11} = p_i + 1, q_i + 1, \tau_i; R$ and $U_{22} = p_i + 2, q_i + 2, \tau_i; R$

4. Aleph-function of two variables

If $r = 2$, we obtain the Aleph-function of two variables defined by K.Sharma [5], and we have the following general expansion formula.

$$\sum_{k=1}^{\infty} \frac{\prod_{j=1}^{A'} (a'_j)_k z^k}{\prod_{j=1}^{B'} (b'_j)_k k!} \mathbb{N}_{UC+D,E:W}^{0,n+C:m_1+F_1,n_1+G_1;m_2+F_2,n_2+G_2} \left(\begin{matrix} z_1 \\ \vdots \\ \vdots \\ z_2 \end{matrix} \middle| \begin{matrix} A, (u_j - k; \rho_j^{(1)}, \rho_j^{(2)})_{1,C}, \\ \vdots \\ B, (w_j - k; \epsilon_j^{(1)}, \epsilon_j^{(2)})_{1,E}, \end{matrix} \right) (v_j + k; \sigma_j^{(1)}, \sigma_j^{(2)})_{1,D} : C_1, (g_j^{(1)} - k; \eta_j^{(1)})_{1,G_1}, (h_j^{(1)} + k; \xi_j^{(1)})_{1,H_1}; \dots : D_1, (i_j^{(1)} + k; \mu_j^{(1)})_{1,F_1}, (l_j^{(1)} - k; \nu_j^{(1)})_{1,l_1}; C_2, (g_j^{(2)} - k; \eta_j^{(2)})_{1,G_2}, (h_j^{(2)} + k; \xi_j^{(2)})_{1,H_2} \dots : D_2, (i_j^{(2)} + k; \mu_j^{(2)})_{1,F_2}, (l_j^{(2)} - k; \nu_j^{(2)})_{1,l_2} \Bigg) = \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \psi(s_1, s_2) \prod_{k'=1}^2 \theta_{k'}(s_{k'})$$

$$\Gamma \left(\begin{matrix} A_1(s_1, s_2) \\ \dots \\ B_1(s_1, s_2) \end{matrix} \right) {}_{A'+P}F_{B'+Q} \left(\begin{matrix} (a_{A'}) \\ \dots \\ (b_{B'}) \end{matrix} ; z \right) z_1^{s_1} z_2^{s_2} ds_1 ds_2 \tag{4.1}$$

with the same notations and validity conditions ($r = 2$). We have the two particular cases

$$\begin{aligned} \text{a) } & \sum_{k=0}^{\infty} \frac{(a'_1)_k}{k!} \aleph_{U_{11}:W}^{0,n+1;V} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix} \middle| \begin{matrix} (u_1 - k; \rho_1^{(1)}, \rho_1^{(2)}), A : C \\ \cdot \\ \cdot \\ (w_1 - k; \epsilon_1^1, \epsilon_1^{(2)}), B : D \end{matrix} \right) \\ &= \aleph_{U_{22}:W}^{0,n+2;V} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix} \middle| \begin{matrix} (u_1; \rho_1^{(1)}, \rho_1^{(2)}), (1 + w_1 + a'_1 - u_1; \epsilon_1^{(1)} - \rho_1^{(1)}, \epsilon_1^{(2)} - \rho_1^{(2)}), A : C \\ \cdot \\ \cdot \\ (w_1 + a'_1; \epsilon_1^{(1)}, \epsilon_1^{(2)}), (1 + w_1 - u_1; w_1 + a'_2; \epsilon_1^{(1)} - \rho_1^{(1)}, \epsilon_1^{(2)} - \rho_1^{(2)}), B : D \end{matrix} \right) \end{aligned} \tag{4.2}$$

where $\epsilon_1^{(1)} - \rho_1^{(1)} > 0$ and $\epsilon_1^{(2)} - \rho_1^{(2)} > 0$

$$\begin{aligned} \text{b) } & \sum_{k=0}^{\infty} \frac{(-)^k (a'_1)_k}{k!} \aleph_{U_{11}:W}^{0,n+1;V} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix} \middle| \begin{matrix} (u_1 - k; \rho_1^{(1)}, \rho_1^{(2)}), A : C \\ \cdot \\ \cdot \\ (-1 + a'_1 + u_1 - k; \rho_1^{(1)}, \rho_1^{(2)}), B : D \end{matrix} \right) \\ &= \aleph_{U_{22}:W}^{0,n+2;V} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix} \middle| \begin{matrix} (u_1; \rho_1^{(1)}, \rho_1^{(2)}), \left(\frac{u_1-1}{2}; \frac{\rho_1^{(1)}}{2}, \frac{\rho_1^{(2)}}{2}\right), A : C \\ \cdot \\ \cdot \\ (1 + u_1; \rho_1^{(1)}, \rho_1^{(2)}), \left(\frac{u_1+1}{2}; +a'_1; \frac{\rho_1^{(1)}}{2}, \frac{\rho_1^{(2)}}{2}\right), B : D \end{matrix} \right) \end{aligned} \tag{4.3}$$

Remark :

If $\tau_i = \tau_i^{(1)} = \tau_i^{(2)} = 1$, the Aleph-function of two variables degenerate to the I-function of two variables defined by Sharma et al [4]. For more details, see the paper of Agrawal et al [1].

6. Conclusion

In this paper, we have established a general expansion formula involving the multivariable Aleph-function by using the generalized hypergeometric function. Due to general nature of the multivariable aleph-function and expansion formula involving here, our formula is capable to be reduced into many known and news expansions involving the special functions of one and several variables.

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