Certain integrals for multivariable Aleph-function involving Jacobi polynomial and kampe de Feriet function

$F.Y. AYANT^1$

1 Teacher in High School , France

ABSTRACT
In this document, we obtain certains integrals and Fourier series expansions involving the multivariable Aleph-function, Jacobi polynomials and Kampe de Feriet function which are the sufficiently general in nature and are capable of yielding a large number of simpler and useful results merely by specializing the parameters in them. Further we establish some special cases.

KEYWORDS : Aleph-function of several variables, integrals, Fourier serie, Jacobi polynomials, Kampe de Feriet function.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

The Kampé de Fériet hypergeometric function will represented as follows.

$$K_{G;H;H'}^{E;F;F'}\left(\begin{array}{c}(\text{e});(\text{f});(\text{f'});\text{x}\\ \\ (\text{g});(\text{h});(\text{h'});\text{y}\end{array}\right)=\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}\frac{\prod_{k=1}^{E}(e_k)_{r+t}\prod_{k=1}^{F}(f_k)_r\prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t}\prod_{k=1}^{H}(h_k)_r\prod_{k=1}^{H'}(h'_k)_t}\times\frac{x^ry^t}{r!t!} \quad (1.1)$$

For further detail see Appell and Kampé de Fériet [1]. For brevity , we shall use the following notations.

$$\epsilon=\frac{\prod_{k=1}^{E}(e_k)_{r+t}\prod_{k=1}^{F}(f_k)_r\prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t}\prod_{k=1}^{H}(h_k)_r\prod_{k=1}^{H'}(h'_k)_t}$$

$$\epsilon_1=\frac{\prod_{k_1=1}^{E_1}(e_{1k_1})_{r_1+t_1}\prod_{k_1=1}^{F_1}(f_{1k_1})_{r_1}\prod_{k_1=1}^{F'_1}(f'_{1k_1})_{t_1}}{\prod_{k_1=1}^{G_1}(g_{1k_1})_{r_1+t_1}\prod_{k_1=1}^{H_1}(h_{1k_1})_{r_1}\prod_{k_1=1}^{H'_1}(h'_{1k_1})_{t_1}}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\epsilon_n=\frac{\prod_{k_n=1}^{E_n}(e_{nk_n})_{r_n+t_n}\prod_{k_n=1}^{F_n}(f_{nk_n})_{r_n}\prod_{k_n=1}^{F'_n}(f'_{nk_n})_{t_n}}{\prod_{k_n=1}^{G_n}(g_{nk_n})_{r_n+t_n}\prod_{k_n=1}^{H_n}(h_{nk_n})_{r_n}\prod_{k_n=1}^{H'_n}(h'_{nk_n})_{t_n}}$$

The Aleph-function of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [3]. The generalized multivariable I-function is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right)$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}] \quad ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\cdots\cdots\cdots\cdots \quad ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{matrix}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{matrix}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\phi_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

with $\omega=\sqrt{-1}$

For more details, see Ayant [2].The reals numbers τ_i are positives for $i=1,\cdots,R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1,\cdots,R^{(k)}$ .The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{n}\alpha_j^{(k)}-\tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0 \quad \text{, with } k=1 \text{ to } r \text{ , } i=1 \text{ to } R \text{ , } i^{(k)}=1 \text{ to } R^{(k)} \tag{1.3}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We will use these following notations in this paper

$$U=p_i,q_i,\tau_i;R\ ;\ V=m_1,n_1;\cdots;m_r,n_r \tag{1.4}$$

$$\mathrm{W}=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.5}$$

$$A=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.6}$$

$$B=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.7}$$

$$C_1=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,$$

$$C_r=\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.8}$$

$$D_1=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,$$

$$D_r=\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.9}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r)=\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{matrix}\middle|\begin{matrix}\mathrm{A}\ ;\ \mathrm{C}_1;\cdots;C_r\\ \cdots\\ \mathrm{B}\ ;\ \mathrm{D}_1;\cdots;D_r\end{matrix}\right) \tag{1.10}$$

We shall note $C=C_1;\cdots;C_r;D=D_1;\cdots;D_r$ (1.11) we have $\aleph(z_1,\cdots,z_r)=\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} z_1 & \text{A ; C} \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ z_r & \text{B ; D}\end{array}\right)$ (1.12)

2. Formula

In this section, we shall consider two general integrals involving the Aleph function of several variables, Jacobi polynomial and Kampe de Feriet functions.

Lemme

$$\int_0^{t_1}\cdots\int_0^{t_r}\prod_{i=1}^r (t_i-x_i)^{\rho_i}x_i^{\sigma_i}P_{n_i}^{(\alpha_i,\beta_i)}(1-y_ix_i)\mathrm{d}x_1\cdots\mathrm{d}x_r=\prod_{i=1}^r\frac{\Gamma(\sigma_i+1)\Gamma(\alpha_i+n_i+1)\Gamma(\rho_i+1)}{n_i\Gamma(\alpha_i+1)\Gamma(\rho_i+\sigma_i+2)}$$

$$_3F_2\left(-n_i,n_i+\alpha_i+\beta_i+1,\sigma_i+1;\alpha_i+1,\rho_i+2;\frac{1}{2}y_it_i\right) \tag{2.1}$$

where $min\{Re(\rho_i,\sigma_i,\alpha_i,\beta_i)\}>0, i=1,\cdots,r$

Proof

We evaluate the t_r-integral with the help of the Erdelyi's formula ([3],p.192,eq.46). Repeating this process $r-1$ times, we obtain (2.1).

We note : $U_{21}=p_i+2,q_i+1,\tau_i;R$

Theorem 1

$$\int_0^t (t-x)^\rho x^\sigma P_n^{(\alpha,\beta)}(1-zx)K_{G;H;H'}^{E;F;F'}\left(\begin{array}{c}\text{(e) ; (f) ; (f') ; k}x^m\\ \\ \text{(g) ; (h) ; (h') ; l}x^m\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\text{z}_1(t-x)^{k_1}x^{l_1}\\ \cdots\\ \cdots\\ \text{z}_r(t-x)^{k_r}x^{l_r}\end{array}\right)\mathrm{d}x$$

$$=t^{\rho+\sigma+1}\sum_{r=0}^\infty\sum_{t=0}^\infty\sum_{u=1}^n\epsilon\Gamma(\alpha+n+1)(-n)_u\left(\frac{zt}{2}\right)^u\frac{(\alpha+\beta+n+1)_u}{n!u!\Gamma(\alpha+u+1)}t^{m(r+t)}k^rl^t$$

$$\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c}\text{z}_1t^{h_1+k_1} & (\text{-}\rho;k_1,\cdots,k_r),(-\sigma-mr-mt-u;l_1,\cdots,l_r),A:C\\ \cdots & \cdots\\ \cdots & \\ \text{z}_rt^{h_r+k_r} & (\text{ (-1 -}\rho-\sigma-u-mr-mt;k_1+l_1,\cdots,k_r+l_r),B:D\end{array}\right) \tag{2.2}$$

provided that

a) $min\{\rho,\sigma,k_i,l_i\}>0, i=1,\cdots,r$;

b) $Re[\rho+\sum_{i=1}^r k_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$; $Re[\sigma+\sum_{i=1}^r l_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.3)

Proof

To prove (2.2), we express the Kampe de Feriet function in serie with the help of (1.1) and the multivariable Aleph-function in the integrand by Mellin-Barnes contour integral (1.2). Changing the order of summations and integrations (which is permissible under the conditions stated above). Finally we calculate the inner integral using the result given by Erdelyi [([3],p.192,eq.46)] ,express the hypergeometric function in serie and interpreting the resulting Mellin-Barnes contour integral as an Aleph-function of several variables. The desired formula is obtained.

Let : $W_{21} = p_{i^{(1)}} + 2, q_{i^{(1)}} + 1, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}} + 2, q_{i^{(r)}} + 1, \tau_{i^{(r)}}; R^{(r)}$

and $V + 2 = m_1, n_1 + 2; \cdots; m_r, n_r + 2$

Theorem 2

$$\int_0^{t_1} \cdots \int_0^{t_r} \prod_{i=1}^{r} (t_i - x_i)^{\rho_i} x_i^{\sigma_i} \;\; P_{p_i}^{(\alpha_i,\beta_i)}(1 - y_i x_i)\, K_{G_i;H_i;H_i'}^{E_i;F_i;F_i'}\left(\begin{array}{c} (\mathrm{e}_i), (f_i), (f_i'); kx_i^{\mathfrak{h}_i} \\ \\ (\mathrm{g}_i), (h_i), (h_i'); lx_i^{\mathfrak{h}_i} \end{array}\right)$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1 (t_1 - x_1)^{k_1} x_1^{l_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r (t_r - x_r)^{k_r} x_r^{l_r} \end{array}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_r = \prod_{i=1}^{r} t_i^{\rho_i+\sigma_i+1} \sum_{s_1,s_1',\cdots,s_r,s_r'=0}^{\infty} \sum_{u_1=0}^{p_1} \cdots \sum_{u_r=0}^{p_r} \prod_{i=1}^{r} \epsilon_i$$

$$k^{s_i} l^{s_i'} \prod_{i=1}^{r} \Gamma(\alpha_i + p_i + 1)(-p_i)_{u_i} \left(\frac{y_i t_i}{2}\right)^{u_i} \frac{(\alpha_i + \beta_i + p_i + 1)_{u_i}}{p_i! u_i! \Gamma(\alpha_i + u_i + 1)} k_i^{s_i} l_i^{s_i'} \aleph_{U:W_{21}}^{0,\mathfrak{n}:V+2}\left(\begin{array}{c|c} \mathrm{z}_1 t_1^{k_1+l_1} & \mathrm{A} : \mathrm{C}_1, \\ \cdot\cdot\cdot & \cdot\cdot\cdot \\ \cdot\cdot\cdot & \\ \mathrm{z}_r t_r^{k_r+l_r} & \mathrm{B} : \mathrm{D}_1, \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho_1; k_1), (-\sigma_1 - \mathfrak{h}_1 s_1 - \mathfrak{h}_1 s_1' - u_1; l_1); \cdots; C_r \quad ,(\text{-}\rho_r; k_r), (-\sigma_r - \mathfrak{h}_r s_r - \mathfrak{h}_r s_r' - u_r; l_r) \\ \cdot\cdot\cdot \qquad\qquad\qquad\qquad \cdot\cdot\cdot \\ (\text{-}1\text{-}\rho_1 - \sigma_1 - \mathfrak{h}_1 s_1 - \mathfrak{h}_1 s_1' - u_1; k_1 + l_1); \cdots; D_r, (\text{-}1\text{-}\rho_r - \sigma_r - \mathfrak{h}_r s_r - \mathfrak{h}_r s_r' - u_r; k_r + l_r) \end{array}\right) \quad (2.3)$$

provided that

a) $min\{\rho_i, \sigma_i, \alpha_i, \beta_i, k_i, l_i\} > 0, i = 1, \cdots, r$

b) $Re[\rho + k_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0\,; Re[\sigma + l_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$ with $i = 1, \cdots, r$

c) $|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is given in (1.3)

Proof

To prove (2.3), we express the product of r Kampe de Feriet functions in series with the help of (1.1) and the multivariable Aleph-function in the integrand by Mellin-Barnes contour integral (1.2). Changing the order of summations and integrations (which is permissible under the conditions stated above). Finally we calculate the inner integral using the result given by the **Lemme,** express the product of r hypergeometric functions in series and interpreting the resulting Mellin-Barnes contour integral as an Aleph-function of several variables. The desired formula is obtained.

3. Expansion formulas

In this section, we consider two expansions of the multivariable Aleph-function and Kampe de Feriet functions in series of Jacobi polynomials. We shall require the orthogonality property of the Jacobi polynomials.

Theorem 3

$$(1-x)^{\rho}x^{\sigma}\,K_{G;H;H'}^{E;F;F'}\left(\begin{array}{c}(\text{ e })\,;\,(\text{ f })\,;\,(\text{ f' })\,;\,\text{kx}^m\\ (\text{ g })\,;\,(\text{ h })\,;\,(\text{ h' })\,;\,\text{lx}^m\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\text{z}_1(1-x)^{k_1}x^{l_1}\\ \cdots\\ \cdots\\ \text{z}_r(1-x)^{k_r}x^{l_r}\end{array}\right)$$

$$=\sum_{r,t,n=0}^{\infty}\sum_{u=1}^{n}\epsilon(-n)_u\frac{(\alpha+\beta+2n+1)_u\Gamma(\alpha+\beta+n+u+1)}{u!\Gamma(\alpha+u+1)\Gamma(\beta+n+1)}k^p l^q\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c}\text{z}_1\\ \cdots\\ \cdots\\ \text{z}_r\end{array}\right|$$

$$\left.\begin{array}{c}(\text{-}\rho-\beta;k_1,\cdots,k_r),(-\sigma-\alpha-mr-mt-u;l_1,\cdots,l_r),A:C\\ \cdots\\ (\text{-}1\ \text{-}\rho-\sigma-\alpha-\beta-u-mr-mt;k_1+l_1,\cdots,k_r+l_r),B:D\end{array}\right)P_n^{(\alpha,\beta)}(1-2x) \quad (3.1)$$

with the same notations.

Provided that

a) $min\{\rho,\sigma,k_i,l_i\}>0, i=1,\cdots,r\,;min\{Re(\alpha),Re(\beta)\}>-1$

b) $Re[\rho+\sum_{i=1}^{r}k_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0\,;Re[\sigma+\sum_{i=1}^{r}l_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz_k|<\frac{1}{2}\left(A_i^{(k)}\pi-k_i-l_i\right)$, where $A_i^{(k)}$ is given in (1.3)

Proof

To prove (3.1), let $f(x)=(1-x)^{\rho}x^{\sigma}K_{G;H;H'}^{E;F;F'}\left(\begin{array}{c}(\text{ e })\,;\,(\text{ f })\,;\,(\text{ f' })\,;\,\text{kx}^m\\ (\text{ g })\,;\,(\text{ h })\,;\,(\text{ h' })\,;\,\text{lx}^m\end{array}\right)$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\text{z}_1(1-x)^{k_1}x^{l_1}\\ \cdots\\ \cdots\\ \text{z}_r(1-x)^{k_r}x^{l_r}\end{array}\right)=\sum_{n=0}^{\infty}C_nP_n^{(\alpha,\beta)}(1-2x) \quad (3.2)$$

where $f(x)$ is a continuous function and bounded variation with interval $(0,1)$. Now , multiplied by $P_u^{(\alpha,\beta)}(1-2x)$ both sides in (3.2) and integrating it with respect x from 0 to 1. Use the orthogonality of Jacobi polynomials, we find

$$C_n=\frac{n(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\int_0^1(1-x)^{\rho+\alpha}x^{\sigma+\beta}K_{G;H;H'}^{E;F;F'}\left(\begin{array}{c}(\text{ e })\,;\,(\text{ f })\,;\,(\text{ f' })\,;\,\text{kx}^m\\ (\text{ g })\,;\,(\text{ h })\,;\,(\text{ h' })\,;\,\text{lx}^m\end{array}\right)$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\text{z}_1(1-x)^{k_1}x^{l_1}\\ \cdots\\ \cdots\\ \text{z}_r(1-x)^{k_r}x^{l_r}\end{array}\right)\text{d}x \quad (3.3)$$

Use the **theorem 1** with $t=1$and $z=2$, we get the value C_n. Finally substituting the value of C_n in the equation (3.2), we get the desired result.

Theorem 4

$$\prod_{i=1}^{r}(1-x_i)^{\rho_i}x_i^{\sigma_i}\,K^{E_i;F_i;F_i'}_{G_i;H_i;H_i'}\left(\begin{array}{c}(\mathrm{e}_i),(f_i),(f_i');kx_i^{\mathfrak{h}_i}\\ \\ (\mathrm{g}_i),(h_i),(h_i');lx_i^{\mathfrak{h}_i}\end{array}\right)\aleph^{0,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c}\mathrm{z}_1(1-x_1)^{k_1}x_1^{l_1}\\ \cdots\\ \cdots\\ \mathrm{z}_r(1-x_r)^{k_r}x_r^{l_r}\end{array}\right)$$

$$=\sum_{s_1,s_1',p_1,\cdots,s_r,s_r',p_r=0}^{\infty}\sum_{u_1=0}^{p_1}\cdots\sum_{u_r=0}^{p_r}\prod_{i=1}^{r}k^{p_i}l^{q_i}\Gamma(\alpha_i+\beta_i+2p_i+1)\frac{(\alpha_i+\beta_i+p_i+u_i)}{u_i!\Gamma(\alpha_i+u_i+1)\Gamma(\beta_i+p_i+1)}$$

$$(-p_i)_{u_i}k_i^{s_i}l_i^{t_i}\aleph^{0,\mathfrak{n}:V+2}_{U:W_{21}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdots\\ \cdots\\ \mathrm{z}_r\end{array}\middle|\begin{array}{l}\mathrm{A}:\mathrm{C}_1,\ (\text{-}\rho_1;k_1),(-\sigma_1-h_1s_1-\mathfrak{h}_1s_1'-u_1;l_1);\cdots;C_r\\ \quad\cdots\qquad\qquad\qquad\qquad\cdots\\ \mathrm{B}:\mathrm{D}_1,(\text{-1-}\rho_1-\sigma_1-h_1s_1-\mathfrak{h}_1s_1'-u_1;k_1+l_1);\cdots;D_r\end{array}\right.$$

$$\left.\begin{array}{l},(\text{-}\rho_r;k_r),(-\sigma_r-\mathfrak{h}_rs_r-\mathfrak{h}_rs_r'-u_r;l_r)\\ \qquad\qquad\cdots\\ ,(\text{-1-}\rho_r-\sigma_r-\mathfrak{h}_rs_r-\mathfrak{h}_rs_r'-u_r;k_r+l_r)\end{array}\right)\prod_{i=1}^{r}P_{p_i}^{(\alpha_i,\beta_i)}(1-2x_i) \tag{3.4}$$

Provided that

a) $min\{\rho_i,\sigma_i,\alpha_i,\beta_i,k_i,l_i\}>0, i=1,\cdots,r\,;min\{Re(\alpha),Re(\beta)\}>-1$

b)$Re[\rho+k_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0\,;Re[\sigma+l_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0\,,i=1,\cdots,r$

c)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.3)

Proof $Let f(x_1,\cdots,x_r)=\prod_{i=1}^{r}(1-x_i)^{\rho_i}x_i^{\sigma_i}\ K^{E_i;F_i;F_i'}_{G_i;H_i;H_i'}\left(\begin{array}{c}(\mathrm{e}_i),(f_i),(f_i');kx_i^{\mathfrak{h}_i}\\ \\ (\mathrm{g}_i),(h_i),(h_i');lx_i^{\mathfrak{h}_i}\end{array}\right)$

$$\aleph^{0,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c}\mathrm{z}_1(1-x_1)^{k_1}x_1^{l_1}\\ \cdots\\ \cdots\\ \mathrm{z}_r(1-x_r)^{k_r}x_r^{l_r}\end{array}\right)=\sum_{p_1,\cdots,p_r=0}^{\infty}C_{p_1,\cdots,p_r}\prod_{i=1}^{r}P_{p_i}^{(\alpha_i,\beta_i)}(1-2x_i) \tag{3.5}$$

The equation (3.5) is valid since $f(x_1,\cdots,x_r)$ is continuous and bounded variation in the domain $(0,1)\times\cdots\times(0,1)$. Multiplying both sides of (3.4) by $\prod_{i=1}^{r}P_{q_i}^{(\alpha_i,\beta_i)}(1-2x_i)$ and integrating with respect to $x_1,\cdots,x_r$ from 0 to 1 $r-times$. Use the orthogonality of Jacobi polynomials $r-times$, we find

$$C_{p_1,\cdots,p_r}=\prod_{i=1}^{r}\frac{p_i(\alpha_i+\beta_i+2p_i+1)\Gamma(\alpha_i+\beta_i+p_i+1)}{\Gamma(\alpha_i+p_i+1)\Gamma(\beta_i+p_i+1)}\int_0^1\cdots\int_0^1\prod_{i=1}^{r}(1-x_i)^{\rho_i}x_i^{\sigma_i}$$

$$P_{p_i}^{(\alpha_i,\beta_i)}(1-2x_i)\ K_{G_i;H_i;H'_i}^{E_i;F_i;F'_i}\left(\begin{matrix}(\mathrm{e}_i),(f_i),(f'_i);kx_i^{\mathfrak{h}_i}\\ \\ (\mathrm{g}_i),(h_i),(h'_i);lx_i^{\mathfrak{h}_i}\end{matrix}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1(1-x_1)^{k_1}x_1^{l_1}\\ \cdots\\ \cdots\\ \mathrm{z}_r(1-x_r)^{k_r}x_r^{l_r}\end{matrix}\right)\mathrm{d}x_1\cdots\mathrm{d}x_r$$

Use the **theorem 2** with $t_i=1$and $y_i=2$, we get the value $C_{p_1,\cdots,p_r}$. Finally substituting the value of $C_{p_1,\cdots,p_r}$ in the equation (3.4), we get the desired result.

4. Multivariable I-function

If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$, the Aleph-function of several variables degenere to the I-function of several variables. The Fourier expansions have been derived in this section for multivariable I-functions defined by Sharma et al [4].

Corollary 1

$$(1-x)^{\rho}x^{\sigma}\ K_{G;H;H'}^{E;F;F'}\left(\begin{matrix}(\mathrm{e});(\mathrm{f});(\mathrm{f'});\mathrm{k}x^{m}\\ \\ (\mathrm{g});(\mathrm{h});(\mathrm{h'});\mathrm{l}x^{m}\end{matrix}\right)I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1(1-x)^{k_1}x^{l_1}\\ \cdots\\ \cdots\\ \mathrm{z}_r(1-x)^{k_r}x^{l_r}\end{matrix}\right)$$

$$=\sum_{r,t,n=0}^{\infty}\sum_{u=1}^{n}\epsilon(-n)_u\frac{(\alpha+\beta+2n+1)_u\Gamma(\alpha+\beta+n+u+1)}{u!\Gamma(\alpha+u+1)\Gamma(\beta+n+1)}k^p l^q I_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix}\mathrm{z}_1\\ \cdots\\ \cdots\\ \mathrm{z}_r\end{matrix}\ \middle|\right.$$

$$\left.\begin{matrix}(\text{-}\rho-\beta;k_1,\cdots,k_r),(-\sigma-\alpha-mr-mt-u;l_1,\cdots,l_r),A:C\\ \cdots\\ (\text{-}1\ \text{-}\rho-\sigma-\alpha-\beta-u-mr-mt;k_1+l_1,\cdots,k_r+l_r),B:D\end{matrix}\right)P_n^{(\alpha,\beta)}(1-2x) \tag{4.1}$$

with the same notations and conditions that (3.1)

Corollary 2

$$\prod_{i=1}^{r}(1-x_i)^{\rho_i}x_i^{\sigma_i}\ K_{G_i;H_i;H'_i}^{E_i;F_i;F'_i}\left(\begin{matrix}(\mathrm{e}_i),(f_i),(f'_i);kx_i^{\mathfrak{h}_i}\\ \\ (\mathrm{g}_i),(h_i),(h'_i);lx_i^{\mathfrak{h}_i}\end{matrix}\right)\ I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1(1-x_1)^{k_1}x_1^{l_1}\\ \cdots\\ \cdots\\ \mathrm{z}_r(1-x_r)^{k_r}x_r^{l_r}\end{matrix}\right)$$

$$=\sum_{s_1,s'_1,p_1,\cdots,s_r,s'_r,p_r=0}^{\infty}\ \sum_{u_1=0}^{p_1}\cdots\sum_{u_r=0}^{p_r}\prod_{i=1}^{r}k^{p_i}l^{q_i}\Gamma(\alpha_i+\beta_i+2p_i+1)\frac{(\alpha_i+\beta_i+p_i+u_i)}{u_i!\Gamma(\alpha_i+u_i+1)\Gamma(\beta_i+p_i+1)}$$

$$(-p_i)_{u_i}k_i^{s_i}l_i^{t_i}I_{U:W_{21}}^{0,\mathfrak{n}:V+2}\left(\begin{matrix}\mathrm{z}_1\\ \cdots\\ \cdots\\ \mathrm{z}_r\end{matrix}\ \middle|\ \begin{matrix}\mathrm{A}:\mathrm{C}_1,\ (\text{-}\rho_1;k_1),(-\sigma_1-h_1s_1-\mathfrak{h}_1s'_1-u_1;l_1);\cdots;C_r\\ \cdots\qquad\cdots\\ \mathrm{B}:\mathrm{D}_1,(\text{-}1\text{-}\rho_1-\sigma_1-h_1s_1-\mathfrak{h}_1s'_1-u_1;k_1+l_1);\cdots;D_r\end{matrix}\right.$$

$$\left.\begin{matrix},(\text{-}\rho_r;k_r),(-\sigma_r-\mathfrak{h}_rs_r-\mathfrak{h}_rs'_r-u_r;l_r)\\ \cdots\\ ,(\text{-}1\text{-}\rho_r-\sigma_r-\mathfrak{h}_rs_r-\mathfrak{h}_rs'_r-u_r;k_r+l_r)\end{matrix}\right)\prod_{i=1}^{r}P_{p_i}^{(\alpha_i,\beta_i)}(1-2x_i) \tag{4.2}$$

with the same notations and conditions that (3.4)

5. Aleph-function of two variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [5]. In this section , we have the following expansion formulas concerning the Aleph-function of two variables.

Corollary 3

$$(1-x)^{\rho}x^{\sigma}\,K_{G;H;H'}^{E;F;F'}\left(\begin{array}{c}(\text{ e });(\text{ f });(\text{ f' });\text{ k}x^{m}\\ \\ (\text{ g });(\text{ h });(\text{ h' });\text{ l}x^{m}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\text{z}_1(1-x)^{k_1}x^{l_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \text{z}_2(1-x)^{k_2}x^{l_2}\end{array}\right)$$

$$=\sum_{r,t,n=0}^{\infty}\sum_{u=1}^{n}\epsilon(-n)_u\frac{(\alpha+\beta+2n+1)_u\Gamma(\alpha+\beta+n+u+1)}{u!\Gamma(\alpha+u+1)\Gamma(\beta+n+1)}k^pl^q\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c}\text{z}_1\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \text{z}_2\end{array}\right|$$

$$\left.\begin{array}{c}(\text{-}\rho-\beta;k_1,k_2),(-\sigma-\alpha-mr-mt-u;l_1,l_2),A:C\\ \cdot\cdot\cdot\\ (\text{-}1\ \text{-}\rho-\sigma-\alpha-\beta-u-mr-mt;k_1+l_1,k_2+l_2),B:D\end{array}\right)P_n^{(\alpha,\beta)}(1-2x) \quad (5.1)$$

with the same notations and conditions that (3.1)

Corollary 2

$$\prod_{i=1}^{2}(1-x_i)^{\rho_i}x_i^{\sigma_i}\,K_{G_i;H_i;H_i'}^{E_i;F_i;F_i'}\left(\begin{array}{c}(\text{ e}_i),(f_i),(f_i');kx_i^{\mathfrak{h}_i}\\ \\ (\text{ g}_i),(h_i),(h_i');lx_i^{\mathfrak{h}_i}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\text{z}_1(1-x_1)^{k_1}x_1^{l_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \text{z}_2(1-x_2)^{k_2}x_2^{l_2}\end{array}\right)$$

$$=\sum_{s_1,s_1',p_1,s_2,s_2',p_2=0}^{\infty}\sum_{u_1=0}^{p_1}\sum_{u_2=0}^{p_2}\prod_{i=1}^{2}\Gamma(\alpha_i+\beta_i+2p_i+1)\frac{(\alpha_i+\beta_i+p_i+u_i)}{u_i!\Gamma(\alpha_i+u_i+1)\Gamma(\beta_i+p_i+1)}$$

$$(-p_i)_{u_i}k_i^{s_i}l_i^{t_i}\aleph_{U:W_{21}}^{0,\mathfrak{n}:V+2}\left(\begin{array}{c}\text{z}_1\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \text{z}_2\end{array}\right|\begin{array}{c}\text{A : C}_1,\ (\text{-}\rho_1;k_1),(-\sigma_1-\mathfrak{h}_1s_1-\mathfrak{h}_1s_1'-u_1;l_1);C_2\\ \cdot\cdot\cdot\qquad\qquad\cdot\cdot\cdot\\ \text{B : D}_1,(\text{-}1\text{-}\rho_1-\sigma_1-\mathfrak{h}_1s_1-\mathfrak{h}_1s_1'-u_1;k_1+l_1);D_2\end{array}$$

$$\left.\begin{array}{c},(\text{-}\rho_;k_2),(-\sigma_2-\mathfrak{h}_2s_2-\mathfrak{h}_2s_2'-u_2;l_2)\\ \cdot\cdot\cdot\\ ,(\text{-}1\text{-}\rho_2-\sigma_2-\mathfrak{h}_2s_2-\mathfrak{h}_2s_2'-u_2;k_2+l_2)\end{array}\right)\prod_{i=1}^{2}P_{p_i}^{(\alpha_i,\beta_i)}(1-2x_i) \quad (6.2)$$

with the same notations and conditions that (3.4)

7. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function ,multivariable Fox's H-function, Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

References

[1] Appel P. and Kampé de Fériet J. Fonctions hypergéométriques et hyperspheriques ; Polynômes D'hermite , Gauthier-Villars , Paris . 1926

[2] Ayant F.Y. An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT). 2016 Vol 31 (3), page 142-154.

[3] Erdelyi A. et al . Table of integral transform, Vol2, McGraw-Hill, New York (1954).

[4] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[5] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page1-13.

Personal adress : 411 Avenue Joseph Raynaud
Le parc Fleuri , Bat B
83140 , Six-Fours les plages
Tel : 06-83-12-49-68
Department : VAR
Country : FRANCE