ALMOST LIFITING MODULES

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ABSTRACT

Let R be a commutative ring with identity and M be an R-module. In this paper we introduce almost lifiting module as a generalization of lifiting modules. The module M is called almost lifiting R-module if for every submodule N of M, there exist submodules A and B such that $ann_M(\, ann_R(N)\,) = A\oplus B$ with $A \subseteq N$ and $B\cap N << M$. Many results about these concepts are introduced and some relationships between these modules and other related modules are studied.

Key words : almost lifiting modules, lifiting modules, comultiplication modules.

1. INTRODUCTION

Throughout this paper R denotes a commutative ring with identity and modules are unitary R-modules. An R-mdule M is called lifiting module if for every submodule N of M, there exists a decomposition of M with $M=A\oplus B$, $A \leq N$ and $B\cap N << B$, equavalently, for every submodule N of M can be written as $N=D\oplus S$, where D is a direct summand of M and $S << M$ [8].

The main goal of this research is to introduce and study the concept almost lifiting modules as a generalization of lifiting modules. An R-module M is called almost lifiting if for every submodule N of M, there exist submodules A and B such that $ann_M(\, ann_R(N)\,) = A\oplus B$ with $A \subseteq N$ and $B\cap N << M$.

The work consists of two sections. In Section two, we supply some examples and properties of almost lifiting modules. We show that the lifiting modules are almost lifiting modules but the converse is not true (Remarks and Examples 2.2 (1) and (2)). Also we see that the almost lifiting property is inherited from a module to each of its direct summands (Proposition 2.3). Among other results, we provide some results looking for under what conditions the almost lifiting modules are lifiting (Proposition

2.7, Proposition 2.8 and Corollary 2.9). Further, and some connections between the almost lifiting modules and other related modules such as multiplication modules comultiplication modules, and projective modules will be studied.

2. Almost Lifiting Modules

In this section we study the concept of almost lifiting modules. Basic facts of this type of modules are investigated. We begin with the following definition.

Definition 2.1. An R-module M is called almost lifiting module if for every submodule N of M, there exist submodules A and B such that $ann_M\,(\,ann_R\,(N)\,) = A \oplus B$ with $A \subseteq N$ and $B \cap N << M$. A ring R is called almost lifiting if R is almost lifiting module as R-module.

Remarks and Examples 2.2.

(1) Every lifiting module is almost lifiting.

Proof. Let M be lifiting R-module. Let N be a submodule M, then M has a decomposition with $M{=}A \oplus B$, $A \subseteq N$ and $B \cap N << M$. Then we have $ann_M\,(ann_R\,(\,N\,)) = ann_M\,(ann_R\,(\,N\,)\,) \cap (\,A \oplus B\,)$. Since $A \subseteq N \subseteq ann_M\,(ann_R\,(\,N\,))$, then by modular law, $ann_M\,(ann_R\,(\,N\,)\,) = A \oplus (\,ann_M\,(ann_R\,(\,N\,)) \cap B\,)$. It follows that $(\,ann_M\,(ann_R\,(\,N\,)) \cap B \cap N) \subseteq (\,B \cap N) << M$. Thus $ann_M\,(ann_R\,(N)) \cap B \cap N << M$. Hence M is almost lifiting.

(2) In general, almost lifiting module need not be lifiting. For example, let M denote $\mathbb{Z}$-module $\mathbb{Z}_8 \oplus \mathbb{Z}_2$. By simple calculation, one can easily see that for every submodule N of M, there exist submodules A and B such that $ann_M\,(\,ann_R\,(N)\,) = A \oplus B$ with $A \subseteq N$ and $B \cap N << M$. But M is not lifiting module.

(3) Every semisimple (or hollow) module is almost lifiting, where an R-module M is called semisimple if every submodule of M is a direct summand [9]. And an an R-module M is called hollow if every proper submodule is small in M [6].

Proof. Since every semisimple (or hollow) module is lifiting, so it is almost lifiting.

(4) If N is a semimaximal submodule of an R-module M ; that is M/N is semisimple R-module [8] then by Remark (3), M/N is almost lifiting.

(5) Every comultiplication module is almost lifiting, where an R-module M is called comultiplication if for every submodule N of M, $ann_M(ann_R(N)) = N$ [2].

Proof. Let M be a comultiplication module, then for every submodule N of M, $ann_M(ann_R(N)) = N = N \oplus \langle 0 \rangle$. So we have $N \subseteq N$ and $\langle 0 \rangle \cap N \ll M$.

In particular for each positive integer n, the $\mathbb{Z}$-module $\mathbb{Z}_n$ is comultiplication and hence it is almost lifiting.

(6) If M is an almost lifiting module then M may not be comultiplication. For example, let M denote $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is semisimple, so it is an almost lifiting. Now, let $N = \langle (\bar{1}, \bar{0}) \rangle$ be the submodule generated by$(\bar{1}, \bar{0})$ implies that $ann_M(ann_{\mathbb{Z}}(N)) = ann_M(2\mathbb{Z}) = M = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \neq N$. Thus M is not comultiplication.

(7) $\mathbb{Z} \oplus \mathbb{Z}_2$ is not almost lifiting $\mathbb{Z}$-module since if $m = (2, \bar{1}) \in \mathbb{Z} \oplus \mathbb{Z}_2$ implies that $ann_{\mathbb{Z} \oplus \mathbb{Z}_2}(ann_{\mathbb{Z}}(m)) = ann_{\mathbb{Z} \oplus \mathbb{Z}_2}(0) = \mathbb{Z} \oplus \mathbb{Z}_2$ where $(\mathbb{Z} \oplus 0)$, $(0 \oplus \mathbb{Z}_2)$ the only direct summands such that $\mathbb{Z} \oplus \mathbb{Z}_2 = (\mathbb{Z} \oplus 0) \oplus (0 \oplus \mathbb{Z}_2)$. Now, we see that $\mathbb{Z} \oplus 0 \nsubseteq \langle (2, \bar{1}) \rangle = 2\mathbb{Z} \oplus \mathbb{Z}_2$. If we write the equality as $\mathbb{Z} \oplus \mathbb{Z}_2 = (0 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z} \oplus 0)$ implies that $(0 \oplus \mathbb{Z}_2) \subseteq \langle (2, \bar{1}) \rangle = 2\mathbb{Z} \oplus \mathbb{Z}_2$. But $(\mathbb{Z} \oplus 0) \cap (2\mathbb{Z} \oplus \mathbb{Z}_2) = 2\mathbb{Z} \oplus 0$ which is not small in $\mathbb{Z} \oplus \mathbb{Z}_2$ because $(2\mathbb{Z} \oplus 0) + (3\mathbb{Z} \oplus \mathbb{Z}_2) = \mathbb{Z} \oplus \mathbb{Z}_2$ where $3\mathbb{Z} \oplus \mathbb{Z}_2$ is a proper submodule of $\mathbb{Z} \oplus \mathbb{Z}_2$. Thus $\mathbb{Z} \oplus \mathbb{Z}_2$ is not almost lifiting as $\mathbb{Z}$-module.

Proposition 2.3. Every direct summand of an almost lifiting module is almost lifiting.

Proof. Let M be an almost lifiting R-module and K be a direct summand of M. Then $M = K \oplus L$ for some submodule L. One can easily see that, every submodule N of M, $ann_M(ann_R(N)) = ann_K(ann_R(N)) \oplus ann_L(ann_R(N))$. Assume that N is a submodule of K, then N is a submodule of M. Since M generalized lifiting, then $ann_M(ann_R(N)) = ann_K(ann_R(N)) \oplus ann_L(ann_R(N)) = A \oplus B$ where A and B submodules of M with $A \subseteq N$ and $B \cap N \ll M$. This implies that $A \subseteq K$ and $B \cap N \ll K$ because K is a direct summand of M. So for every submodule N of K, there exists a submodule A of K such that $ann_K\,(\,ann_R\,(N)\,) = A = A\oplus <0>$ where $A \subseteq N$ and $<0> \cap N = <0> << N$. Hence K is an almost lifiting R-module.

Example 2.4. Let $M = \mathbb{Q} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module. Then M is not almost lifiting, since if so, it follows that the submodule $\mathbb{Q}$ is almost lifiting which is a contradiction.

Proposition 2.5. Let M be an almost lifiting R-module. If $ann_M\,(\,ann_R\,(N)\,)$ is a direct summand of M for every submodule N of M. Then M is a lifiting module.

Proof. Let N be a submodule of M. Since M is an almost lifiting module, then $N \subseteq ann_M\,(\,ann_R\,(N)\,) = A\oplus B$ with $A \subseteq N$ and $B \cap N << M$. By modular law, we have $N = N\cap(\,A\oplus B)$. Hence $N = A\oplus(N\cap B)$. Since $ann_M\,(\,ann_R\,(N)\,)$ is a direct summand of M, then $M = ann_M\,(\,ann_R\,(N)\,)\oplus D$ for some submodule D of M. This implies that $M = A\oplus B\oplus D$. Hence $N = A\oplus(N\cap B)$, where A is a direct summand of M and $B \cap N << M$. Therefore M is a lifiting module.

Let R be an integral domain, an R-module M is called divisible if $rM = M$ for every non-zero element $r \in R$ [7].

Proposition 2.6. Let M be an almost lifiting module over an integral domain R . If every submodule N of M is divisible, then M is a lifiting module.

Proof. Let N be a submodule of M. Since M is almost lifiting, then $ann_M\,(\,ann_R\,(N)\,) = A\oplus B$ with $A \subseteq N$ and $B \cap N << M$. By hypothesis, N is divisible, implies that $ann_R\,(N) = 0$. Hence $M = ann_M\,(\,ann_R\,(N)\,) = A\oplus B$ with $A \subseteq N$ and $B \cap N << M$. Therefore M is lifiting.

Recall that a submodule B of an R-module M is called a relative complement of A if B is a maximal submodule with respect to the preoperty $A\cap B=0$. And a submodule A of an R-module M is called an essential of M (or M is an essential extension of A) if $A \cap B \neq 0$, for every submodule B of M. If A has no proper essential extension in M, then A is said to be closed [4]. Let M be an R-module, the singular submodule of M is $Z\,(\,M\,) = \{\, m \in M \mid Im = 0$ for some essential ideal I of $R\,\}$. If $Z\,(\,M\,) = 0$, then M is called nonsingular [4].

Proposition 2.7. Let M be an almost lifiting nonsingular R-module. If for each submodule N of M, $ann_M(ann_R(N))$ has a decomposition $A\oplus B$ such that B is a relative complement of A. Then M is a lifiting module.

Proof. Let N be a submodule of M. Since M is almost lifiting, then $ann_M(ann_R(N)) = A\oplus B$ with $A \subseteq N$ and $B \cap N << M$. By hypothesis, B is a relative complement of A. So by [4], $A\oplus B$ is an essential submodule of M. Thus $ann_M(ann_R(N))$ is essential in M. We claim that $ann_M(ann_R(N))$ is a closed submodule in M. To show that. Suppose that $ann_M(ann_R(N))$ is an essential submodule in K for some submodule K of M and $ann_M(ann_R(N)) \neq K$, then there is $0 \neq m \in K$, $m \notin ann_M(ann_R(N))$ and an essential ideal I (may be R) of R such that $0 \neq m I \subseteq ann_M(ann_R(N))$. Hence $ann_R(N)\, m\, I = 0$. Since M is nonsingular, it follows that $ann_R(N)\, m = 0$. This implies that $m \in ann_M(ann_R(N))$ which is a contradiction. So $ann_M(ann_R(N)) = A\oplus B$ is closed in M. But $A\oplus B$ is essential in M. Therefore $M = A\oplus B$ with $A \subseteq N$ and $B \cap N << M$. Thus M is lifiting.

Recall that an R-module M is called prime if $ann_R(M) = ann_R(N)$ for ever nonzero submodule N of M [3].

Proposition 2.8. If M is an almost lifiting prime R-module, then M is lifiting.

Proof. Let N be a submodule of M. As M is prime R-module, then $ann_R(M) = ann_R(N)$. It follows that $M = ann_M(ann_R(M)) = ann_M(ann_R(N))$. But M is almost lifiting, so $ann_M(ann_R(N)) = A\oplus B$ for some $A \subseteq N$ and $B \cap N << M$. Thus $M = A\oplus B$, $A \subseteq N$ and $B \cap N << B$. Hence M is lifiting.

Recall that an R-module M is called quasi-Dedekind if for every $f \in End_R(M)$, $\ker f = 0$ [10]. It is well-known that every quasi-Dedekind module is prime, so we have the following result

Corollary 2.9. If M is an almost lifiting quasi-Dedekind module, then M is lifiting.

Proposition 2.10. Let M be an R-module such that for each submodule N of M, there exists a decomposition $M=A\oplus S$ where $ann_M(ann_R(N)) \cap S << M$ and $A \subseteq N$. Then M is almost lifiting.

Proof. Since $M=A\oplus S$, then ann_M (ann_R (N) = $(A\oplus S)$ ∩ ann_M (ann_R (N) . Then by modular law, ann_M (ann_R (N) = $A \oplus$ [ann_M (ann_R (N) $\cap S$], because $A \subseteq N \subseteq ann_M$ (ann_R (N). Say $B=$ ann_M (ann_R (N) $\cap S$. As $B \ll M$ by hypothesis. Therefore ann_M (ann_R(N) = $A\oplus B$ with $A \subseteq N$ and $B \cap N \ll M$. Thus M is almost lifiting.

Lemma 2.11. Let M and M' be an R-modules and $f : M \longrightarrow M'$ be an isomorphism. If M is almost lifiting, then so is M'.

Proof. Let N' be a submodule of M'. Then $f(N) = N'$ for some submodule N of M. Since M is almost lifiting, implies that ann_M (ann_R(N) = $A\oplus B$ for some $A \subseteq N$ and $B\cap N \ll M$. It is easily to show that $f(ann_M$ (ann_R(N)) = $ann_{M'}$ (ann_R($f(N)$) and $f(A\oplus B) = f(A) \oplus f(B)$. This implies that $ann_{M'}$ (ann_R($f(N)$) = $f(A) \oplus f(B)$. Further $f(A) \subseteq f(N) = N'$ and $f(B) \cap f(N) = f(B\cap N) \ll f(M) = M'$. Thus M' is almost lifiting.

Proposition 2.12. Let R be a ring . Then every free R-module is almost lifiting if and only if every projective R-module is almost lifiting.

Proof. ($\Rightarrow$) Let M be a projective R-module, then there exists a free R-module F and an epimorphism $f : F \longrightarrow M$. We have the following short exact sequence $0 \longrightarrow \ker f \xrightarrow{i} \mathrm{F} \xrightarrow{f} M \longrightarrow 0$. Since M is a projective then the sequence splits and hence $\mathrm{F} \cong \ker f \oplus M$. By our assumption, F is almost lifiting, implies that every direct summand is almost lifiting by Proposition (2.3). But M is isomorphic to a direct summand of F. So by Lemma (2.11), M is almost lifiting.

($\Leftarrow$) It is clear.

Using a similar argument one can show the following.

Proposition 2.13. Let R be a ring . Then every finitely generated free R-module is almost lifiting if and only if every finitely generated projective R-module is almost lifiting.

Remark 2.14. We claim that the direct sum of two almost lifiting modules need not be almost lifiting, but we have no example to ensure this. However the next results present a certain condition under which a direct sum of almost lifiting modules is again almost lifiting.

Recall that a submodule A of an R-module M is called fully invariant if $f(A) \subseteq A$, for every endomorphism f of M [10]. If every submodule of M is fully invariant, then M is called duo module [10].

Proposition 2.15. Let $M = M_1 \oplus M_2$ be R-module such that M is duo module. If M_1 and M_2 are almost lifiting R-modules, then so is M.

Proof. Let N be a submodule of M. Since N is fully invariant, then by [12, Lemma 2.1.], $N = (N \cap M_1) \oplus (N \cap M_1)$. Put $N \cap M_1 = N_1$ and $N \cap M_2 = N_2$. Since M_1 and M_2 are almost lifiting, then $ann_{M_1}(ann_R(N_1)) = A_1 \oplus B_1$ for some $A_1 \subseteq N_1$ and $B_1 \cap N_1 << M_1$ and $ann_{M_2}(ann_R(N_2)) = A_2 \oplus B_2$ for some $A_2 \subseteq N_2$ and $B_2 \cap N_2 << M_2$. Now we have $ann_M(\mathrm{ann}_R(N) = ann_M(\mathrm{ann}_R(N_1 \oplus N_2)) \cap M = [ann_M(\mathrm{ann}_R(N_1) \oplus ann_M(\mathrm{ann}_R(N_2)] \cap (M_1 \oplus M_2) = [ann_M(\mathrm{ann}_R(N_1) \cap M_1] \oplus [ann_M(\mathrm{ann}_R(N_2) \cap M_2] = ann_M(\mathrm{ann}_R(N_1) \oplus ann_M(\mathrm{ann}_R(N_2) = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2) = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$. But $A_1 \subseteq N_1$ and $A_2 \subseteq N_2$, implies that $A_1 \oplus A_2 \subseteq N$. Since $B_1 \cap N_1 \subseteq B_1 \cap M_1$ and $B_2 \cap N_2 \subseteq B_2 \cap M_2$, implies that $B_1 \cap M_1 << M_1$ and $B_2 \cap M_2 << M_2$. Hence $(B_1 \oplus B_2) \cap (M_1 \oplus M_2) = (B_1 \cap M_1) \oplus (B_2 \cap M_2) << M_1 \oplus M_2 = M$. Therefore M is almost lifiting.

Before we give the next result, we need to state known the following lemma

Lemma 2.16. Let M_1 and M_2 be R_1 and R_2-modules respectively. If $M = M_1 \oplus M_2$ and $R = R_1 \oplus R_2$.Then

(1) $\mathrm{ann}_R(N_1 \oplus N_2) = ann_{R_1}(N_1) \oplus ann_{R_2}(N_2)$ for any submodules N_1 in M_1 and N_2 in M_2.

(2) $\mathrm{ann}_M(I_1 \oplus I_2) = ann_{M_1}(I_1) \oplus ann_{M_2}(I_2)$ for any ideals I_1 in R_1 and I_2 in R_2.

Proposition 2.17. Let M_1 and M_2 be almost lifiting as R_1 and R_2-modules respectively such that for each submodule N in $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$ for some submodule N_1 in M_1 and N_2 in M_2. Then M is an almost lifiting module.

Proof. Let N be a submodule of M and $R = R_1 \oplus R_2$. By hypothesis, $N = N_1 \oplus N_2$ for some submodule N_1 in M_1 and N_2 in M_2. Since M_i is an almost lifiting R_i-module where (i = 1, 2) then $ann_{M_1}(ann_{R_1}(N_1)) = A_1 \oplus B_1$ for some $A_1 \subseteq N_1$ and $B_1 \cap N_1 << M_1$ and $ann_{M_2}(ann_{R_2}(N_2)) = A_2 \oplus B_2$ for some $A_2 \subseteq N_2$ and $B_2 \cap N_2 << M_2$. By lemma 2.16, $ann_M(ann_R(N)) = ann_M(ann_{R_1}(N_1) \oplus ann_{R_2}(N_2)) = ann_{M_1}(ann_{R_1}(N_1)) \oplus ann_{M_2}(ann_{R_2}(N_2)) = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2) = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$. Since $A_1 \subseteq N_1$ and $A_2 \subseteq N_2$, then $A_1 \oplus A_2 \subseteq N_1 \oplus N_2 = N$. Also $B_1 \cap N_1 << M_1$ and $B_2 \cap N_2 << M_2$, so $(B_1 \oplus B_2) \cap (N_1 \oplus N_2) = (B_1 \cap N_1) \oplus (B_2 \cap N_2) << M_1 \oplus M_2 = M$. Hence M is almost lifiting.

Proposition 2.18. Let M_1 and M_2 be comultiplication R-modules such that $ann_R(M_1) + ann_R(M_2) = R$. Then $M = M_1 \oplus M_2$ is almost lifiting.

Proof. Let N be a submodule of M. By hypothesis, $ann_R(M_1) + ann_R(M_2) = R$ then by a part of the proof of [1, Proposition (4.2),CH.1], any submodule of $M = M_1 \oplus M_2$ can be written as a direct sum of two submodules of M_1 and M_2. Thus $N = N_1 \oplus N_2$ for some submodules N_1 and N_2 of M_1 and M_2 respectively. Since M_1 and M_2 are comultiplication, it follows that $N = ann_{M_1}(ann_R(N_1)) \oplus ann_{M_2}(ann_R(N_2))$. On the other hand by Remark and Example 2.2 (5), M_1 and M_2 are almost lifiting. So there exist submodules $A_1 \subseteq M_1$, $A_2 \subseteq M_2$ such that $ann_{M_1}(ann_R(N_1)) = A_1 \oplus B_1$, $A_1 \subseteq N_1$, $B_1 \cap N_1 << M_1$ and $ann_{M_2}(ann_R(N_2)) = A_2 \oplus B_2$, $A_2 \subseteq N_2$, $B_2 \cap N_2 << M_2$. Hence $N = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2)$ where $A_1 \oplus A_2 \subseteq N_1 \oplus N_2 = N$ and $(B_1 \oplus B_2) \cap (N_1 \oplus N_2) = (B_1 \cap N_1) \oplus (B_2 \cap N_2) << M_1 \oplus M_2 = M$. Hence M is almost lifiting.

Recall that a submodule K of an R-module M is called a quasi-hollow if for each submodules A and B of M with $K = A + B$, then either $K = A$ or $K = B$ [7].

Proposition 2.19. Let M be an almost lifiting R-module such that $J(M) = 0$. If every nonzero submodule of M is quasi-hollow. Then M is a comultiplication module.

Proof. Since M is almost lifiting, then for every submodule N of M, there exist submodules A and B such that $N \subseteq ann_M(ann_R(N)) = A \oplus B$ with $A \subseteq N$ and $B \cap N << M$. By hypothesis either $ann_M(ann_R(N)) = A \subseteq N$. If $ann_M(ann_R(N)) = B$, then

$B \cap N = N$. Since J(M) = 0, it follows that N = 0 which is a contradiction. Therefore M is comultiplication.

Recall that an R-module M is called multiplication if for each submodule N of M there exists an ideal I of R such that $N = IM$ [5].

Proposition 2.20. Let M be a multiplication finitely generated faithful R-module. Then M is almost lifiting if and only if R is almost lifiting.

Proof. ($\Rightarrow$) Let I be an ideal of R. Then $N = IM$ is a submodule of M, so that $ann_M(\mathrm{ann}_R(N)) = A \oplus B$ for some $A \subseteq N$ and $B \cap N \ll M$. Since M is a multiplication, $A = JM$, $B = KM$ for some ideals J and K of R. Then we have $ann_M(\mathrm{ann}_R(N)) = ann_M(\mathrm{ann}_R(IM)) = ann_M(\mathrm{ann}_R(I)) = ann_R(\mathrm{ann}_R(\mathrm{I}))M$. This implies $ann_R(\mathrm{ann}_R(\mathrm{I}))M = JM \oplus KM = (J \oplus K)M$. But M is multiplication finitely generated faithful, so that $ann_R(\mathrm{ann}_R(\mathrm{I})) = J \oplus K$. On the other hand $JM \subseteq N = IM$, then $J \subseteq I$. Further, $KM \cap IM = (K \cap I)M \ll RM$, implies that $K \cap I \ll R$. Thus R is almost lifiting.

($\Leftarrow$) Let N be a submodule of M. Since M is a multiplication, $N = IM$ for some ideal I of R. But R is almost lifiting , then $ann_R(\mathrm{ann}_R(\mathrm{I})) = J \oplus K$ for some $J \subseteq I$ and $K \cap I \ll R$. So that $ann_M(\mathrm{ann}_R(N)) = ann_M(\mathrm{ann}_R(IM)) = ann_M(\mathrm{ann}_R(I)) = ann_R(\mathrm{ann}_R(\mathrm{I}))M = (J \oplus K)M = JM \oplus KM = A \oplus B$. Since $J \subseteq I$, implies that $A = JM \subseteq IM = N$. Further, $K \cap I \ll R$, and M is a multiplication finitely generated faithful then $(K \cap I)M \ll M$. Since M is faithful and multiplication, so $B \cap N = KM \cap IM = (K \cap I)M$. Thus $B \cap N \ll M$ and hence M is almost lifiting.

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