Possibility measure space

FirasHusseanMaghool

AL- Qadisiya University

Faculty of Computer science and information technology

Department *of mathematics*

AL- Qadisiya , Iraq

Abstract:

In this paper, we defineapossibility measureas a set function $Pos : F \to [0,1]$ where F is σ-field and discuss some properties of possibility measure to prove important theory

Keyword: σ-field, measurable space , possibility measure.

1-**Introduction:**

Possibility measures were introduced by Zadeh [3]in 1978. In his view,Possibility theory is an uncertainty theory devoted to the handling of incomplete information.

Possibility theory lies at the crossroads between fuzzy sets, probability and non-monotonic reasoning. Qualitative possibility theory is closely related to belief revision theory. Possibilistic logic provides a rich representation setting,which enables the handling of lower bounds of possibility theory measures, while remaining close to classical logic. Quantitative possibility theory is the simplestframework for statistical reasoning with imprecise probabilities.

possibility and necessity measures can also be the basis of a full-edged representation ofpartial belief that parallels probability[2],[4].

In this work define the possibility measure as a set function and study some properties of this measure.

2-Preliminaries

Definition(2.1)[1]:A family F of subsets of a set Ω is called a σ-field on a set Ω, if

(1) $\Omega \in F$

(2) If $A \in F$, then $A^c \in F$

(3) If $A_n \in F, n = 1,2,\ldots$ then $\bigcup_{n=1}^{\infty} A_n \in F$

A measurable space is a pair (Ω,F),where Ω is a set and F is σ-field on Ω

Asubset A of Ω is called measurable (measurable with respect to the σ-field F) , if $A \in F$ i.e. any member of F is called a measurable set.

Definition(2.2)[4]:

Let(Ω,F) be a measurable space, a set function $Pos : F \to [0,1]$ is said to be possibility measure if it is satisfies the following axioms:

(1) $Pos(\Omega) = 1$

(2) $Pos(\phi) = 0$

(3) For every sequence $\{A_n\}$ in F, we have

$$\theta(\bigcup_{n=1}^{\infty} A_n) = \max_{1 \le n \le \infty} \{Pos(A_n)\}$$

A Possibility space is a tripe (Ω, F, Pos) where Ω is a set, F is a σ - field, Pos a possibility measure on F.

3-Main result

Theorem(3.1)

Let (Ω, F, Pos) bea possibility space, then

(1) $0 \le Pos(A) \le 1$.

(2) $Pos(A_1 \bigcup A_2) \le Pos(A_1) + Pos(A_2)$.

(3) If $A_1, A_2 \in F$ and $A_1 \subset A_2$ then $Pos(A_1) \le Pos(A_2)$..

(4) If $A_1, A_2 ..., A_n \in F$, then $Pos(\bigcup_{k=1}^{n} A_k) \le \max_{1 \le k \le n} \{Pos(A_k)\}$.

(5) $Pos(\bigcup_{k=1}^{\infty} A_k) \le \sum_{i=1}^{\infty} Pos(A_k)$

(6) $Pos(A_1 \bigcap A_2) \le \min\{Pos(A_1), Pos(A_2)\}$

(7) $Pos(A_1) + Pos(A_2) - 1 \le Pos(A_1 \bigcap A_2)$

(8) If $A_1, A_2 \in F$, then $Pos(A_1 - A_2) \ge Pos(A_1) - Pos(A_2)$.

Proof :

(1) Let $A \in F$

since $\phi \subseteq A \subseteq \Omega$ then

$Pos(\phi) \le Pos(A) \le Pos(\Omega)$.

(2) $Pos(A_1 \bigcup A_2) = Max\{Pos(A_1), Pos(A_2)\}$
$\le Pos(A_1) + Pos(A_2)$.

(3) If $A_1, A_2 \in F$, $A_1 \subset A_2 \in F$ then

$A_1^c \cup A_2 = \Omega$

$Pos(A_1^c \cup A_2) \le Pos(A_1^c) + Pos(A_2)$

$\Rightarrow Pos(\Omega) \le Pos(A_1^c) + Pos(A_2)$

$\Rightarrow 1 \le Pos(A_1^c) + Pos(A_2)$

$\Rightarrow 1 \le 1 - Pos(A_1) + Pos(A_2)$

$\Rightarrow Pos(A_1) \le Pos(A_2)$.

(4) put $A_K = \phi$ for all $k \ge n$

$\Rightarrow$

$Pos(\bigcup_{k=1}^{n} A_K) = Pos(\bigcup_{k=1}^{\infty} A_k) = \max_{1 \le k \le \infty} \{Pos(A_k)$

Since $Pos(A_k) = Pos(\phi) = 0$ *when* $k \ge n$

then

$\max_{1 \le k \le \infty} \{Pos(A_k)\} = \max_{1 \le k \le n} \{Pos(A_k)\}$

$\Rightarrow Pos(\bigcup_{k=1}^{n} A_k) \le \max_{1 \le k \le n} \{Pos(A_k)\}$.

(5) $Pos(\bigcup_{k=1}^{\infty} A_k) = \max_{1 \le k \le \infty} \{Pos(A_k)$

$\le Pos(A_1) + Pos(A_2) + ...$

$\le \sum_{k=1}^{\infty} Pos(A_k)$

(6) Since $A_1 \cap A_2 \subset A_1$ and $A_1 \cap A_2 \subset A_2$

$\Rightarrow Pos(A_1 \cap A_2) \subset Pos(A_1)$ and

$Pos(A_1 \cap A_2) \subset Pos(A_2)$

$Po(A_1 \cap A_2) \le \min\{Pos(A_1), Pos(A_2)\}$

(7) Since

$$Pos(A_1^c \cup A_2^c) \le Pos(A_1^c) + Pos(A_2^c)$$

$$-Pos(A_1^c \cup A_2^c) \ge -(Pos(A_1^c) + Pos(A_2^c))$$
$$\ge -(1 - Pos(A_1)) - (1 - Pos(A_2))$$
$$\ge Pos(A_1) + Pos(A_2) - 2$$

And

$$Pos(A_1 \cap A_2) = 1 - Pos(A_1 \cap A_2)^c$$
$$= 1 - Pos(A_1^c \cup A_2^c) \ge 1 + Pos(A_1) + Pos(A_2) - 2$$
$$Pos(A_1 \cap A_2) \ge Pos(A_1) + Pos(A_2) - 1$$

Then

$$Pos(A_1) + Pos(A_2) - 1 \le Pos(A_1 \cap A_2)$$

From definition (2.2) we have $\max\{Pos(A_1), Pos(A_2)\} = Pos(A_1 \cup A_2)$

since

$$\max\{Pos(A_1), Pos(A_2)\} \le Pos(A_1) + pos(A_2)$$

then $Pos(A_1 \cup A_2) \le Pos(A_1) + Pos(A_2)$

(8) If

$$Pos(A_1 - A_2) = pos(A_1 \cap A_2^c) \ge Pos(A_1) + pos(A_2^c) - 1$$

(from part (5) $= pos(A_1) + 1 - Pos(A_2) - 1$

$$= Pos(A_1) - Pos(A_2)$$

$$Pos(A_1 - A_2) \ge Pos(A_1 \cap A_2{}^c) \ge Pos(A_1) + Pos(A_2^c) - 1$$
$$\ge Pos(A_1) + (1 - Pos(A_2)) - 1$$

Then

$$Pos(A_1 - A_2) \ge Pos(A_1) - Pos(A_2).$$

Definition (3.2)[1] :

A sequence $\{A_n\}$ of subset of a set Ω is said to be increasing if $A_n \subset A_{n+1}$ for $n = 1,2,\ldots$. And is said to be decreasing if $A_{n+1} \subset A_n$ for $n = 1,2,\ldots$.

A monotone sequence of sets is one which either increasing or decreasing

Theorem(3.3)[1]:

Any monotone sequence from subset in Ω is converge.

Definition (3.4)[1]:

If $\{A_n\}$ is an increasing sequence of subset of a set Ω and $\bigcup_{n=1}^{\infty} A_n = A$,we say that A_n an increasing sequence of a set with limit *A*, or that A_n increase to *A*, write $A_n \uparrow A$, also if $\{A_n\}$ is a decreasing sequence of subset of a set Ω and $\bigcap_{n=1}^{\infty} A_n = A$, we say that the A_n a decreasing sequence of a set with limit *A*, or that the A_n decrease to *A*, write $A_n \downarrow A$.

Theorem (3.5)[1]:

Let $\{A_n\}$ be a sequence of subset of a set Ω and let $A \subset \Omega$

(1) If $A_n \uparrow A$ then $A_n^c \downarrow A^c$

(2) If $A_n \downarrow A$ then $A_n^c \uparrow A^c$

Theorem(3.6):

Let (Ω, F, Pos) be a Possibility space, and let $\{A_n\}$ be a sequence in *F* with $Pos(A_n) \to 0$ as $n \to \infty$, then for any $A \in F$, we have

$$\lim_{n\to\infty} Pos(A \cup A_n) = \lim_{n\to\infty} Pos(A - A_n) = Pos(A)$$

Proof :

Since $A \subset A \cup A_n \Rightarrow Pos(A) \subset Pos(A \cup A_n)$ for each n, and since

$$Pos(A \cup A_n) \leq Pos(A) + Pos(A_n) \quad \text{for each } n,$$

it follows that

$$Pos(A) < Pos(A \cup A_n) \leq Pos(A) + Pos(A_n)$$

for each n.

we have $Pos(A \cup A_n) \to Pos(A)$ by using $Pos(A_n) \to 0$.

since $(A - A_n) \subset A \subset (A - A_n) \cup A_n)$ we have

$$Pos(A - A_n) \leq Pos(A) \subset Pos(A - A_n) + Pos(A_n)$$

Then $Pos(A - A_n) \to Pos(A)$ by using $Pos(A_n) \to 0$.

Theorem (3.7):

Let (Ω, F, Pos) be a Possibility space and let $\{A_n\}$ be a sequence in F, we have $\lim_{n\to\infty} Pos(A_n) > 0$ if $A_n \uparrow \Omega$ and $\lim_{n\to\infty} Pos(A_n) < 1$ if $A_n \downarrow \phi$

Proof :

Let $A_n \uparrow \Omega$ and $\Omega = \bigcup_{n=1}^{\infty} A_n$,

$$1 = Pos(\Omega) = Pos(\bigcup_{n=1}^{\infty} A_n) = \sup Pos(A_n) \leq \sum_{n=1}^{\infty} Pos(A_n)$$

Since $Pos(A_n)$ is increasing to n, we have $\lim_{n\to\infty} Pos(A_n) > 0$

If $A_n \downarrow \phi$, then $A_n^c \uparrow \Omega$

$$\lim_{n\to\infty} Pos(A_n) = 1 - \lim_{n\to\infty} Pos(A_n^C) < 1$$

Theorem(3.8):

Let (Ω, F, Pos) be a Possibility space, for any sequence $\{A_n\}$ in F, we have

(1) If $A_n \uparrow A$ then $\lim_{n\to\infty} Pos(A_n) \leq Pos(A)$.

(2) If $A_n \downarrow A$ then $\lim_{n\to\infty} Pos(A_n) \geq Pos(A)$.

Proof:

(1) Since $A_n \uparrow A \Rightarrow A_n \subset A_{n+1}$ for $n = 1,2,\ldots$ and $\bigcup_{n=1}^{\infty} A_n = A$

$\Rightarrow A_n \subset A$ for $n = 1,2,\ldots \Rightarrow Pos(A_n) \leq Pos(A)$.

(2) Since $A_n \downarrow A \Rightarrow A_n^c \uparrow A^c$

$$\Rightarrow \lim_{n\to\infty} Pos(A_n^C) \leq Pos(A^C)$$

$$\Rightarrow \lim_{n\to\infty} (1 - Pos(A_n)) \leq 1 - Pos(A)$$

$$\lim_{n\to\infty} Pos(A_n) \geq Pos(A).$$

Theorem(3.9):

Let (Ω, F, Pos) be a Possibility space and let $\{A_n\}$ is a sequence of subsets of Ω, If

$\sum_{n=1}^{\infty} Pos(A_n) < \infty$ then

$Pos\{\lim_{n\to\infty} \sup_n A_n\} = 0$.

Proof:

Since

$$\lim_{n\to\infty} \sup A_n = \bigcap_{n=1}^{\infty} \bigcup_{K=n}^{\infty} A_K = \lim_{n\to\infty} \bigcup_{k=n}^{\infty} A_K$$

$\Rightarrow Pos(\lim_{n\to\infty} \sup A_n) \le Pos(\bigcup_{k=n}^{\infty} A_K)$ for all n

$$\Rightarrow Pos(\lim_{n\to\infty} \sup A_n) \le \sum_{k=n}^{\infty} Pos(A_k) \to 0 \quad as\, n \to \infty$$

Definition (3.10):

Let (Ω, F, Pos) be a Possibility space .we say that Pos is

(1) Continuous from above at $A \in F$, if $\lim_{n\to\infty} Pos(A_n) = Pos(A)$, for $\{A_n\}$ be a sequence in F with $A_n \downarrow A$.

(2) Continuous from below at $A \in F$, if $\lim_{n\to\infty} Pos(A_n) = Pos(A)$, for $\{A_n\}$ be a sequence in F with $A_n \uparrow A$.

Pos is called continuous from above if it is continuous from above at A for all $A \in F$, also Pos is called continuous from below at A for all $A \in F$.

Theorem (3.11):

Let (Ω, F, Pos) be a Possibility space, then the following statements are equivalent:

(1) Pos is continuous from above at $A \in F$.

(2) Pos is continuous from below at $A \in F$.

Proof:

(1)$\Rightarrow$ (2)

Let $\{A_n\}$ be a sequence in F, with $A_n \uparrow A$, we have

$$A_n^c - A^c \downarrow \phi \Rightarrow \lim_{n\to\infty} Pos(A_n^c - A^c)$$
$$= Pos\theta(\phi) = 0$$

$$Pos(A_n^c - A^c) \ge Pos(A_n^c) - \theta(A^c)$$

Since $\Rightarrow \lim_{n\to\infty} Pos(A_n^c - A^c)$

$$\ge \lim_{n\to\infty} Pos(A_n^c) - Pos(A^c)$$

$$\Rightarrow \lim_{n\to\infty} Pos(A_n^c) - Pos(A^c) \le 0$$

$$\Rightarrow 1 - \lim_{n\to\infty} Pos(A_n) - 1 + Pos(A\ ) \le 0$$

$$\Rightarrow \lim_{n\to\infty} Pos(A_n) \ge Pos(A)$$

Since $\lim_{n\to\infty} Pos(A_n) \le Pos(A)$

$$\Rightarrow \lim_{n\to\infty} Pos(A_n) = Pos(A)$$

That is Pos is continuous from below at $A \in F$.

(2) $\Rightarrow$(1)

Let $\{A_n\}$ be a sequence in F, with $A_n \downarrow A$, we have

$A_n^c \cup A \uparrow \Omega \Rightarrow \lim_{n\to\infty} Pos(A_n^c \cup A) = Pos(\Omega) = 1$

Since $Pos(A_n^c \cup A) \le Pos(A_n^c) + Pos(A)$

$\Rightarrow \lim_{n\to\infty} Pos(A_n^c \cup A) \le \lim_{n\to\infty} Pos(A_n^c) + Pos(A)$

$1 \le \lim_{n\to\infty} Pos(A_n^c) + Pos(A)$

$1 \le 1 - \lim_{n\to\infty} Pos(A_n) + Pos(A)$

Since $\lim_{n\to\infty} Pos(A_n) \ge Pos(A)$

$\Rightarrow \lim_{n\to\infty} Pos(A_n) = Pos(A)$

That is Pos is continuous from above at $A \in F$

Reference

[1] R.B .Ash, "Real Analysis and probability", university of lllinois,Academic press(1972).

[2] B. R Gaines and L. Kohout, Possible automata. Proc. Int. Symp. Multiple-Valued logics, Bloomington, IN, pages 183-196, 1975.

[3] L. A. Zadeh, “Fuzzy sets as a basis for a theory of possibility,”Fuzzy Sets and Systems, vol. 1, pp. 3–28, 1978

[4] B. Liu, Uncertainty Theory, 2nd ed., Springer-Verlag, Berlin, 2007.