# e-Semimaximal Submodules

Inaam, M.A.Hadi[#1], Sameeah Hasoon Aidi[*2]
[#1] Department of Mathematics, College of Education for pure Science/ Ibn-Al-Haitham, University of Baghdad, Iraq
*2 The Ministry of Education, General Directorate of Education Rasafa/2

***Abstract***

*In this paper, we introduce and study the concept e-semimaximal submodule. Also many relationships of this concept with other related concepts are given.*

**Key Words:** *e-semimaximal submodule, semimaximal submodules, δ-semimaximal submodule, multiplication modules.*

## 1- INTRODUCTION

Let R be a commutative ring with unity and let M be a left R-module. A proper submodule N of M is called semimaximal if $\frac{M}{N}$ is semisimple [8]. Y-Wang in [10] introduced the concept δ-semimaximal submodule, where a proper submodule N of M is called δ-seminaximal if there exist submodules $N_1, \ldots, N_k$ of M such that $N = \bigcap_{i=1}^{k} N_i$ and $\frac{M}{N_i}$ is singular simple, for each $i = 1, \ldots, k$. Also δ-semimaximal submodules had been studied in [6]. Recall that a submodule W of M is called essential (denoted by $W \underset{e}{\leq} M$) if $N \cap K = (0)$ implies $K = (0)$, where K is a submodule of M [7]. Note that if W = (0) then $W \underset{e}{\leq} M$ if and only if M = (0), [7]. In this paper, we call that a proper submodule N of M is e-semimaximal if $N = \bigcap_{i=1}^{k} N_i$ for some essential maximal submodules $N_1, \ldots, N_k$. We give the main properties for this concept and its relations with other classes of submodules.

## 2- MAIN RESULTS

**Remarks and Examples (2.1):**

**(1)** If N is an e-semimaximal submodule in an R-module M, then N is δ-semimaximal, but the converse is not true in general.

**Proof:** Since N is e-semimaximal, $N = \bigcap_{i=1}^{n} N_i$ for some essential maximal submodules $N_i$, $i = 1, \ldots, n$. Hence $\frac{M}{N_i}$ is simple for all $i = 1, \ldots, n$. By [5,Prop.1.21,p.32], $N_i \underset{e}{\leq} M$ implies $\frac{M}{N_i}$ is singular, $\forall\, i = 1, \ldots, n$. Thus N is δ-semimaximal.

**(2)** If $N \lneqq W \leq M$ and W is e-semimaximal, then it is not necessarily that N is e-semimaximal, for example: in the Z-module $Z_9$, $(\bar{3})$ is e-semimaximal, but $(\bar{0}) < (\bar{3})$ is not e-semimaximal.

**(3)** It is clear that the intersection of two e-semimaximal submodules is e-semimaximal.

**(4)** It is clear that every e-semimaximal submodule is essential and the convery may not be true, for example: in the Z-module $Z_{16}$, $N = <4> \underset{e}{\leq} z_{16}$ but it is not e-semimaximal.

**(5)** A homomorphic image of e-semimaximal submodule need not be e-semimaximal, for example: N = 6Z in the Z-module Z is e-semimaximal. If $\pi: Z \longrightarrow Z/N \simeq Z_6$ then $\pi(N) = (\bar{0})$ which is not e-semimaximal in $Z_6$.

**(6)** Let $f$: $M \longrightarrow M'$ be an R-epimorphism and let W be an e-maximal submodule of M'. Then $f^{-1}(W)$ is an e-maximal submodule of M.

**Proof:** It is easy, so is omitted.

**(7)** Let $f$: $M \longrightarrow M'$ be an R-isomorphism, let N < M. If N is an e-semimaximal submodule of M, then $f(N)$ is an e-semimaximal submodule of M'.

**Proof:**

Since N is e-semimaximal, $N = \bigcap_{i=1}^{n} N_i$ for some essential maximal submodules $N_1, \ldots, N_n$. By essentiality of $N_i$ and the condition $f$ is monomorphism. We have $f(N_i) \underset{e}{\leq} M'$, for each $i = 1, \ldots, n$. Also $f$ is an epimorphism and $N_i$ is maximal imply $f(N_i)$ is maximal in M'. Beside this, $f$ is monomorphism implies that $f(N) = \bigcap_{i=1}^{n} f(N_i)$. Thus $f(N)$ is an e-semimaximal.

**(8)** If N and W are isomorphic submodules of an R-module M such that N is an e-semimaximal, then it is not necessarily that W is e-semimaximal. For example: in the Z-module Z, N = 6Z is e-semimaximal and $N \cong W = 4Z$, but W is not e-semimaximal.

**Proposition (2.2):**
Let M be a uniform R-module, N < M. Then N is e-semimaximal if and only if N is δ-semimaximal.
**Proof:** ⇒ It is clear by Rem. (2.1(1)).
⇐ Since N is δ-semimaximal, $N = \bigcap_{i=1}^{n} W_i$, for some submodules $W_1, \ldots, W_n$ such that $\frac{M}{W_i}$ is singular simple, for each i = 1, …, n. But $\frac{M}{W_i}$ is simple imply $W_i$ is maximal, and since M is uniform, so that $W_i \underset{e}{\leq} M$, ∀ i = 1, …, n. Thus N is e-semimaximal.

**Proposition (2.3):**
Let M be a nonsingular R-module, N < M. Then N is e-semimaximal if and only if N is δ-semimaximal.
**Proof:** ⇒ It follows by Rem. (2.1(1)).
⇐ Since N is δ-semimaximal. $N = \bigcap_{i=1}^{n} W_i$, for some submodules $W_1, \ldots, W_n$ where $\frac{M}{W_i}$ is singular simple, ∀ i = 1, …, n. As $\frac{M}{W_i}$ is simple for each i = 1, …, n, so that $W_i$ is maximal, ∀ i = 1, …, n. On the other hand M is nonsingular and $\frac{M}{W_i}$ is singular imply $W_i \underset{e}{\leq} M$ by [5]. It follows that N is e-semimaximal.

**Corollary (2.4):**
Let M be a nonsingular module over an integral domain, let (0) ≠ N < M. If N is maximal, then N is e-semimaximal.
**Proof:** Since N is maximal, N ≠ (0), then by [6, Th.2.6], N is δ-semimaximal and so by Prop. (2.3), N is e-semimaximal.

Note that the condition M is nonsingular is necessary condition in Cor.2.4, for example in Z-module $Z_6$, (0) ≠ N = $(\bar{2}) < Z_6$ is maximal but not e-semimaximal.

Recall that an R-module M is called fully prime if every proper submodule of M is prime.
Recall that an R-module M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that N = IM. Equivalently M is a multiplication R-module if for each submodule N of M, N = [N:M]M, [3].

**Proposition (2.5):**
Let M be a fully prime multiplication R-module. Then every e-semimaximal submodule is maximal.
**Proof:** Let N be an e-semimaximal submodule. Then $N = \bigcap_{i=1}^{n} N_i$ for some essential maximal submodules $N_1, \ldots, N_n$. By hypothesis, M is fully prime, we have N is a prime submodule. As M is multiplication, we have $N \supseteq N_t$ for some t = 1, …, n. It follows that $N = N_t$ (since $N_t$ is maximal). Thus N is maximal.

**Proposition (2.6):**
Every e-semimaximal submodule N of an R-module M is semimaximal, but not conversely.
**Proof:** As N is e-semimaximal, $N = \bigcap_{i=1}^{n} W_i$ for some essential maximal submodules $W_1, \ldots, W_n$. Thus $\frac{M}{N}$ is isomorphic to a submodule of $\frac{M}{W_1} \oplus \ldots \oplus \frac{M}{W_n}$. But $\frac{M}{W_1} \oplus \ldots \oplus \frac{M}{W_n}$ is semisimple, hence $\frac{M}{N}$ is semisimple. Thus N is semimaximal.

**Example:** $(\bar{0})$ in the Z-module $Z_6$ is semimaximal but it is not e-semimaximal.

Recall that an R-module is F-regular if every submodule N of M is pure; that is IM ∩ N = IN for each ideal I of R, [4].
An R-module M is called fully stable if every submodule N of M is stable; where N is stable means that for each R-homomorphism $f: N \longrightarrow M$, $f(N) \subseteq N$, [1].

**Proposition (2.7):**
Let M be a cyclic R-module such that $ann_R M$ is e-semimaximal. Then M is fully stable.
**Proof:** As $ann_R M$ is e-semimaximal, so by Prop. (2.6) $ann_R M$ is semimaximal. Hence by [8] M is F-regular and so every proper submodule is semiprime. Then by [2, Cor.(4.11),p.66], M is fully stable.

**Remark (2.8):** Let N < M such that $[N \underset{R}{:} M]$ is e-semimaximal. Then $\frac{M}{N}$ is F-regular, where $[N \underset{R}{:} M] = \{r \in R : rM \subseteq N\}$.

**Proof:** By Prop.(2.6), [N:M] is semimaximal. Hence by [8,Prop.(1.3.8], $\frac{M}{N}$ is F-regular R-module.

**Proposition (2.9):** Every e-semimaximal submodule is semiprime.

**Proof:** Let N be an e-semimaximal submodule of an R-module M. Then $N=\bigcap_{i=1}^{n} W_i$ for some essential maximal sumodules $W_1,\ldots,W_n$ of M. But every maximal submodule is prime, so that $W_1,\ldots,W_n$ are prime submodules. Thus N is semiprime [2,Prop.(3.1),p.53].

The converse of Prop.(2.9) need not be true for example: (0) in the Z-module is semiprime but not e-semimaximal.

**Remark (2.10):** Let M be a cyclic R-module. If every proper submodule is e-semimaximal. Then M is fully stable.

**Proof:** By Prop. (2.9), every proper submodule of M is semiprime. Hence M is fully stable by [2,Prop.(4.10),p.66].

**Lemma (2.11):** Let R be a principal ideal domain (PID), let $I<R$, $I\neq(0)$. Then I is a semiprime ideal of R if and only if I is the intersection of finite number of prime ideals.

**Proposition (2.12):**

Let R be a PID., let $I<R$, $I\neq(0)$. Then I is a semiprime ideal if and only if I is an e-semimaximal ideal.

**Proof:** It follows directly by Lemma (2.11) and the fact that every nonzero proper prime ideal of a PID is maximal and every nonzero ideal of R is essential in R.

Note that the condition R is a PID is a necessary condition in Prop. (2.12), for example in the ring $Z_{12}$, $I=<\overline{6}>$ is a semiprime ideal but not e-semimaximal.

**Theorem (2.13):**

Let M be a faithful finitely generated multiplication R-module and let $N<M$. Then the following statements are equivalent:

**(1)** N is an e-semimaximal submodule of M.

**(2)** $[N\underset{R}{:}M]$ is an e-semimaximal ideal of R.

**(3)** N = IM for some e-semimaximal ideal I of R.

**Proof:** (1) $\Rightarrow$ (2) By (1), $N=\bigcap_{i=1}^{n} W_i$ for some essential maximal submodules $W_1,\ldots,W_n$. Then $[N\underset{R}{:}M]=[\bigcap_{i=1}^{n} W_i\underset{R}{:}M]=\bigcap_{i=1}^{n}[W_i\underset{R}{:}M]$ and as $W_i$ is maximal in M, $\forall$ i = 1, …, n, we have $[W_i\underset{R}{:}M]$ is a maximal ideal in R, $\forall$ i = 1, …, n. Also, since M is a faithful multiplication R-module and $N_i\underset{e}{\leq}M$. Hence $\exists\ J_i\underset{e}{\leq}R$ such that $N_i=J_iM$, $\forall$ i = 1, …, n, by [3,Th.(2.13)], so that $J_i=[N_i\underset{R}{:}M]\underset{e}{\leq}R$, $\forall$ i = 1, …, n, [3,Th.(3.1)]. It follows that $[N\underset{R}{:}M]$ is an e-semimaximal ideal of R.

(2) $\Rightarrow$ (3) It is clear since N = $[N\underset{R}{:}M]M$.

(3) $\Rightarrow$ (1) Since N = IM for some e-semimaximal ideal I of R, hence $I=\bigcap_{i=1}^{n} J_i$ for some essential maximal ideals $J_1$, …, $J_n$ of R. Hence $N=(\bigcap_{i=1}^{n} J_i)M$. But M is faithful multiplication, so by [3,Th.(1.6)], $N=\bigcap_{i=1}^{n}(J_iM)$. Also by [3,Th.(3.1)], [3,Th.(2.13)] $J_iM$ is maximal in M and $J_iM\underset{e}{\leq}M$, $\forall$ i = 1, …, n. Thus N is an e-semimaximal submodule of M.

**Corollary (2.14):**

Let M be a finitely generated faithful multiplication module over a PID R, let $(0)\neq N<M$. Then N N is e-semimaximal if and only if N is semiprime.

**Proof:** $\Rightarrow$ It is clear by Prop.(2.9).

$\Leftarrow$ Since N is a semiprime submodule, then $[N\underset{R}{:}M]$ is a semiprime ideal of R. But M is a multiplication R-module and $N\neq(0)$, so that $[N\underset{R}{:}M]\neq(0)$. Hence by Prop. (2.12), $[N\underset{R}{:}M]$ is an e-semimaximal ideal of R. Thus N is an e-semimaximal submodule of M by Th. (2.13).

Recall that the Jacobson radical of an R-module M (denoted by J(M) or Rad M) is the intersection of all maximal submodules of M, if M has maximal submodules and J(M) = M if M has no maximal submodules [7]. Equivalently, $J(M)=\sum_{U\leq M} U$, where U is a small submodules of M, [7]. Also U is a small submodules of M (denoted by $U\ll M$) if U is a proper submodule of M and $U+W\neq M$ for any proper submodule W of M, [7]. D.X.Zhou and X.R.Zhang introduced $Rad_eM'$, where $Rad_eM=\cap\{N\underset{e}{<}M$: N is maximal

in M} if M has maximal submodules and $Rad_eM=M$ if M has no maximal submodule [11].
Equivalently, $Rad_eM = \Sigma N$, where $N \ll_e M$, [11] and $N \ll_e M$ if $N + W = M$ and $W \underset{e}{\leq} M$, implies $W = M$, [11].

Similarly we define the concept e-J(M) (or e-Rad M) as follows: if M has e-semimaximal submodule then e-Rad M $= \cap$ {N:N is an e-semimaximal submodule of M} and e-Rad M = M if M has no e-semimaximal submodule.

However the following proposition shows that $Rad_eM$ and e-Rad M are identical.

**Proposition (2.15):**
For an R-module M, $Rad_eM$ = e-Rad M.

**Proof:** Let $m \in Rad_eM$. Then m belongs to any essential maximal submodule of M, so m belongs to any finite intersection of essential maximal submodule. Hence m belongs to any semimaximal submodule and so $m \in$ e-Rad M; that is $Rad_eM \subseteq$ e-Rad M.
Now let $m \in$ e-Rad M; hence m is in any semimaximal submodule of M. But every essential maximal submodule of M is semimaximal, so that $m \in \cap \{N \underset{e}{<} M$: N is maximal in M$\} = Rad_eM$.
Hence e-Rad M $\subseteq Rad_eM$. Thus e-Rad M = $Rad_eM$.

**Theorem (2.16):**
Let M be a faithful finitely generated multiplication R-module. Then $Rad_eM= (Rad_eR)M$.

**Proof:** Since $Rad_eM$ = e-Rad M, so that e-Rad M = $\cap$ {N:N is semimaximal submodule in M}. But M is a fathful finitely generated multiplication R-module, so every semimaximal submodule N of M, N = IM for some semimaximal ideal I of R by Th. (2.13). Thus e-Rad M = $\cap$ {IM:I is semimaximal idealof R}. But M is faithful multiplication, so that $\cap$(IM) = ($\cap$I)M by [3,Th.(1.6)].
Hence e-Rad M =(e-Rad R)M = $(Rad_eR)M$.

**Theorem (2.17):**
Let M be an R-module. Consider the following statements:

**(1)** M is Artinian.

**(2)** M satisfies descending chain condition on e-small submodules and on e-semimaximal submodules.

**(3)** M satisfies descending chain condition on e-small submodules and $Rad_eM$ is e-semimaximal submodule.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) if $Rad_eM$ is Artinian.

**Proof:** (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Since M satisfies descending chain condition on e-semimaximal, M has a minimal e-semimaximal submodule say N. So $N = \bigcap_{i=1}^{n} N_i$, $N_i$ is essential maximal for each i = 1, …, n. Hence $Rad_eM \leq N$. So if $Rad_eM = M$, then M = N which is a contradiction. Thus $M \neq Rad_e M$. If P is any maximal essential submodule of M, then $N \cap P$ is e-semimaximal submodule. But N is minimal e-semimaximal, so $N = N \cap P$, thus $N \leq P$. Hence $N \leq Rad_e M$. But $Rad_e M \leq N$. So that $N = Rad_e M$. Thus $Rad_e M$ is e-semimaximal submodule.

(3) $\Rightarrow$ (1) $Rad_e M$ is Artinian. As $Rad_e M$ is e-semimaximal, $M \neq Rad_e M$ and $Rad_eM = \bigcap_{i=1}^{n} P_i$ for some $P_i$ essential maximal submodule of M for each i = 1,…, n. But $P_i \underset{e}{\leq} M$, implies $\frac{M}{P_i}$ is singular, [4,Proposition 1.21,p.32]. Also $P_i$ is maximal, implies $\frac{M}{P_i}$ is simple. Since $\frac{M}{Rad_eM} = \frac{M}{\bigcap_{i=1}^{n} P_i}$ submodule of $\frac{M}{P_1} \oplus \ldots \oplus \frac{M}{P_n}$, and $\frac{M}{P_1} \oplus \ldots \oplus \frac{M}{P_n}$ is a semisimple module so $\frac{M}{P_1} \oplus \ldots \oplus \frac{M}{P_n}$ is Artinian. Hence $\frac{M}{Rad_eM}$ is Artinian. But $Rad_e M$ is Artinian, it follows that M is Artinian.

Next we have that:

**Proposition (2.18):**
Let I be an e-semimaximal ideal of a ring R. Then I[X] is an e-semimaximal ideal of R[X], provide that every ideal of R[X] has the form K[X] for some ideal K of R.

**Proof:** Since I is e-semimaximal, $I = \bigcap_{i=1}^{n} J_i$ for some essential maximal ideals $J_1$, …, $J_n$ of R and then $I[X] = (\bigcap_{i=1}^{n} J_i)[X]$. It follows that $I[X] = \bigcap_{i=1}^{n} (J_i[X])$.
But $J_i \underset{e}{\leq} R$, $\forall$ i = 1, …, n implies that $J_i[X] \underset{e}{\leq} R[X]$ by [9,Exc.30,p.116].
On the other hand for any i = 1, …, n, $J_i[X]$ is a maximal ideal in R, since if there exists an ideal $W_i[X]$ in R[X] such that $J_i[X] \leq W_i[X]$ for some ideal W in R. Then $J_i \subseteq W_i$. Hence $J_i$ = W and so $J_i[X]$ = W[X]. Thus I[X] is an e-semimaximal ideal in R[X].

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