Balanced Domination Number of Union of Graphs

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Abstract

Let G= (V, E) be a graph. A Subset D of V is called a dominating set of G if every vertex in V-D is adjacent to atleast one vertex in D. The Domination number γ (G) of G is the cardinality of the minimum dominating set of G. Let G = (V, E) be a graph and let f be a function that assigns to each vertex of V to a set of values from the set {1,2,.......k} that is, f:V(G) → {1,2,.....k} such that for each u,v ∈ V(G), f(u)≠f(v), if u is adjacent to v in G. Then the dominating set D ⊆V (G) is called a balanced dominating set if $\sum_{u\epsilon D} f(u) = \sum_{v\epsilon V-D} f(v)$. *In this paper, we determine the balanced domination number for union of graphs.*

Keywords: *balanced domination number, union, pendant*

Mathematics subject classification: *05C69*

1. INTRODUCTION

Let G = (V, E) be a graph with vertex set V and edge set E. The degree of v denoted by deg_G (v) is the number of vertices adjacent to v in G. A leaf vertex (also pendant vertex) is a vertex with degree one. An edge of a graph is said to be pendant if one of its vertices is a pendant vertex.

Let G = (V, E) be a graph and let f be a function that assigns to each vertex of V to a set of values from the set {1,2,.......k} that is, f:V(G) → {1,2,.....k} such that for each u,v ϵ V(G), f(u)≠f(v), if u is adjacent to v in G. Then the set D ⊆V (G) is called a balanced dominating set if $\sum_{u\epsilon D} f(u) = \sum_{v\epsilon V-D} f(v)$

The balanced domination number $\gamma_{bd}(G)$ is the minimum cardinality of the balanced dominating set.

The set D ⊆ V (G) is called strong balanced dominating set if $\sum_{u\epsilon D} f(u) \geq \sum_{v\epsilon V-D} f(v)$.Also the set D ⊆ V (G) is called weak balanced dominating set if $\sum_{u\epsilon D} f(u) \leq \sum_{v\epsilon V-D} f(v)$

The sum of the values assigned to each vertex of G is called the total value of G. that is, Total value= f (V) = $\sum_{v\epsilon V(G)} f(v)$.

Theorem 1.1

Let G be a graph with n vertices. Then G has a balanced dominating set iff f (V) = $\sum_{v\epsilon V(G)} f(v)$ *is even.*

Proved in [6].

Theorem 1.2

Let G be a graph with n vertices. Then G has no balanced dominating set iff f (V) = $\sum_{v\epsilon V(G)} f(v)$ *is odd.*

Proved in [6].

2. UNION OF GRAPHS

Lemma 2.1

Let G be a graph obtained by attaching a pendant vertex to each of vertices of C_{4n} *then the number of vertices of value 1 is equal to the number of vertices of value 2.*

That is, if n_1 *is the number of vertices of value 1 and* n_2 *is the number of vertices of value 2 in the* γ_{bd} *set of G then* $n_1 = n_2$. *Also* $n_1 = n_2 = 2n$.

Proof:

Let G be a graph obtained by attaching a pendant vertex to each of vertices of C_{4n}.

The cycle C_{4n} has 4n vertices.
Attaching a pendant vertex to each of vertices of C_{4n} , we get 8n vertices.
These 8n vertices divided into 4n vertices of value 1 and 4n vertices of value 2.
Therefore, 4n 1's + 4n 2's =4n+ 2(4n)
$$= 4n + 8n$$
$$= 12n.$$
that is, f(V)= 12n.
Hence $\sum_{v\epsilon D} f(v)$ =6n.
if n_1 is the number of vertices of value 1 and n_2 is the number of vertices of value 2 in the γ_{bd} set of G, then $n_1+2n_2 = 6n$.
In this graph G, we have 4n pendant vertices in which 2n vertices of value 1 and 2n vertices of value 2.
To cover these 2n pendant vertices of value 2, we have to take either the pendant vertex of value 2 or vertex adjacent to that pendant vertex which is of value 1.
Therefore, if m_1 vertices of value 2 are from pendant vertices then we have to take $2n-m_1$ vertices of value 1 other than pendant vertex.
Similarly, To cover these 2n pendant vertices of value 1, we have to take either the pendant vertex of value 1 or vertex adjacent to that pendant vertex which is of value 2.
Therefore, if m_2 vertices of value 1 are from pendant vertices then we have to take $2n-m_2$ vertices of value 2 other than pendant vertex.
Hence $n_1 = 2n-m_1+ m_2$ and
$$n_2 = 2n - m_2 + m_1.$$
we have,
$$n_1+2n_2 = 6n$$
$$2n-m_1+ m_2+2(2n - m_2+ m_1) = 6n$$
$$2n-m_1+ m_2+4n - 2m_2+2 m_1 = 6n$$
$$6n+m_1-m_2=6n$$
$$m_1-m_2=0$$
$$m_1 = m_2.$$
since $m_1 = m_2$, we get $n_1 = n_2$ and $n_1 = n_2 = 2n$.

Theorem 2.2

Let G be a graph obtained by attaching a pendant vertex to each of vertices of C_{4n}. then $\gamma_{bd}(G) = 4n$.

Proof:

Let G be a graph obtained by attaching a pendant vertex to each of vertices of C_{4n}.
The cycle C_{4n} has 4n vertices.
Attaching a pendant vertex to each of vertices of C_{4n} , we get 8n vertices.
These 8n vertices divided into 4n vertices of value 1 and 4n vertices of value 2.
Therefore, 4n 1's + 4n 2's =4n+ 2(4n)
$$= 4n + 8n$$
$$= 12n.$$
that is, f(V)= 12n.
Hence $\sum_{v\epsilon D} f(v)$ =6n.
suppose $n_1+2n_2 = 6n$ where n_1 is the number of vertices of value 1 and n_2 is the number of vertices of value 2.
then $\gamma_{bd}(G) = n_1 + n_2$.
we have to prove $\gamma_{bd}(G) = n_1 + n_2 = 4n$.
we prove this by induction on n.
Let n =1.
we get $n_1 + 2n_2 = 6$
since $2n_2$ is even, n_1 must be even.
therefore, $n_1 = 2$ and $n_2 = 2$.
Hence $\gamma_{bd}(G) = n_1 + n_2 = 4 = 4n$.
Assume that the result is true for n-1.
Let G' be the graph obtained by attaching a pendant vertex to each of vertices of C_{4n-4}.
then G' has 8n-8 vertices and $\sum_{v\epsilon D} f(v)$ =6n-1 =6n-6.
Let m_1 denote the number of vertices of value 1 and m_2 denote the number of vertices of value 2 of the graph G'.
then $\gamma_{bd}(G') = m_1 + m_2 = 4(n-1) = 4n-4$.
we have,
$$m_1+2m_2=6n-6$$
$$m_1 +2m_2+6 = 6n-6+6$$
$$(m_1 +2) +2 (m_2 +2) = 6n \text{ (by Lemma 2.1)}$$
Therefore,
$$\gamma_{bd}(G) = m_1 +2 +m_2+2$$
$$= m_1 +m_2+4$$

= 4n-4+4
=4n.

Example 2.1:

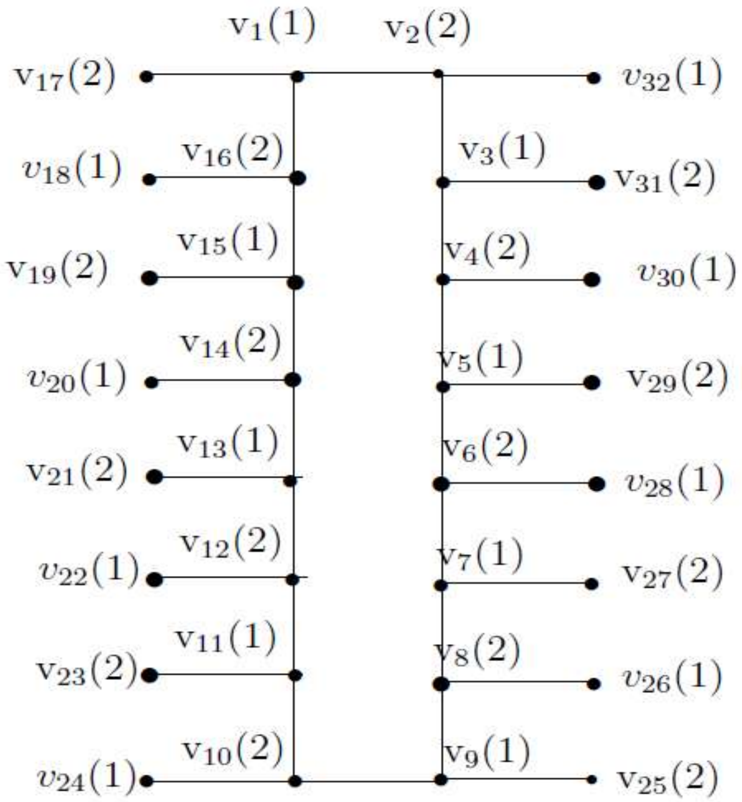

Figure 1

In this graph, f(V)= 48

D= $\{v_1,v_2,v_3,v_4, v_5,v_6,v_7,v_8,v_9,v_{10},v_{11},v_{12},v_{13},v_{14},v_{15} ,v_{16}\}$ is a balanced dominating set.

$\sum_{v \epsilon D} f(v)$= 2+2+2+2+2+2+2+2+1+1+1+1+1+1+1+1= 24

γ_{bd}=16

Lemma 2.3

Let G be a graph obtained by attaching a pendant edge to each vertices of Path P_n then the number of vertices of value 1 is equal to the number of vertices of value 2 in the γ_{bd} set of G.
that is, if n_1 is the number of vertices of value 1 and n_2 is the number of vertices of value 2 in the γ_{bd} set of G then $n_1 = n_2$. Also $n_1 = n_2 = n/2$.

Proof:

Let G be a graph obtained by attaching a pendant edge to each vertices of P_n.
The Path P_n has n vertices.
Attaching a pendant edge to each vertices of P_n , we get 2n vertices.
These 2n vertices divided into n vertices of value 1 and n vertices of value 2.
Therefore, n 1's + n 2's =n+ 2(n)
=3n.
that is, f(V)= 3n.
Hence $\sum_{v \epsilon D} f(v)$ =3n/2.
if n_1 is the number of vertices of value 1 and n_2 is the number of vertices of value 2 in the γ_{bd} set of G, then n_1+2n_2 =3n/2.
In this graph G, we have n pendant vertices in which n/2 vertices of value 1 and n/2 vertices of value 2.
To cover these n/2 pendant vertices of value 2, we have to take either the pendant vertex of value 2 or vertex adjacent to that pendant vertex which is of value 1.
Therefore, if m_1 vertices of value 2 are from pendant vertices then we have to take n/2-m_1 vertices of value 1 other than pendant vertex.

Similarly, To cover these n/2 pendant vertices of value 1, we have to take either the pendant vertex of value 1 or vertex adjacent to that pendant vertex which is of value 2.
Therefore, if m_2 *vertices of value 1 are from pendant vertices then we have to take* $n/2-m_2$ *vertices of value 2 other than pendant vertex.*
Hence $n_1 = n/2 - m_1 + m_2$ *and*

$$n_2 = n/2 - m_2 + m_1.$$

we have,

$$n_1+2n_2 = 3n/2$$
$$n/2 - m_1 + m_2 + 2(n/2 - m_2 + m_1) = 3n/2$$
$$n/2 - m_1 + m_2 + n - 2m_2 + 2m_1 = 3n/2$$
$$3n/2 + m_1 - m_2 = 3n/2$$
$$m_1 - m_2 = 0$$
$$m_1 = m_2.$$

since $m_1 = m_2$*, we get* $n_1 = n_2$ *and* $n_1 = n_2 = n/2$.

Theorem 2.4

Let G be a graph obtained by attaching a pendant edge to each vertices of Path P_n*. then*

$$\gamma_{bd}(G) = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

Proof:

Let G be a graph obtained by attaching a pendant edge to each vertices of Path P_n.
The Path P_n *has n vertices.*
Attaching a pendant edge to each vertices of P_n *, we get 2n vertices.*
These 2n vertices divided into n vertices of value 1 and n vertices of value 2.
Therefore, n *1's +* n *2's* $= n + 2(n) = 3n$.
that is, $f(V) = 3n$.

Hence $\sum_{v\in D} f(v) = 3n/2$.

If n is odd, $\frac{3n}{2}$ *is odd.*

Then $\gamma_{bd}(G) = 0$.

If n is even, suppose $n_1+2n_2 = 3n/2$ *where* n_1 *is the number of vertices of value 1 and* n_2 *is the number of vertices of value 2.*
then $\gamma_{bd}(G) = n_1 + n_2$.
we have to prove $\gamma_{bd}(G) = n_1 + n_2 = n$.
we prove this by induction on n.
Let $n = 2$.
we get $n_1 + 2n_2 = 3$
therefore, $n_1 = 1$ *and* $n_2 = 1$.
Hence $\gamma_{bd}(G) = n_1 + n_2 = 2 = n$.
Assume that the result is true for n-2.
Let G' be the graph obtained by attaching a pendant edge to each vertices of P_{n-2}.
then G' has n-4 vertices and $\sum_{v\in D} f(v) = \frac{3(n-2)}{2} = \frac{3n-6}{2}$

$$= \frac{3n}{2} - 3$$

Let m_1 *denote the number of vertices of value 1 and* m_2 *denote the number of vertices of value 2 of the graph G'.*
then $\gamma_{bd}(G') = m_1 + m_2 = n-2$.

we have,

$$m_1+2m_2 = \frac{3n}{2} - 3$$
$$m_1 + 2m_2 + 3 = \frac{3n}{2} - 3 + 3$$
$$(m_1+1) + 2(m_2+1) = \frac{3n}{2} \text{ (by Lemma 2.3)}$$

Therefore,

$$\gamma_{bd}(G) = m_1 + 1 + m_2 + 1$$
$$= m_1 + m_2 + 2$$
$$= n-2+2$$
$$= n.$$

Example 2.2:

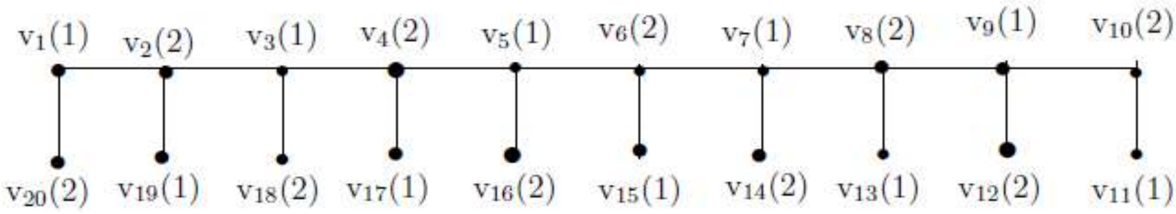

Figure 2

In this graph P_{10} (n is even) with pendant edge at each vertex, f(V)= 30

D= { v_2, v_5,v_8,v_9,v_{11},v_{14},v_{15},v_{17},v_{18},v_{20}} is a balanced dominating set.

$\sum_{v\epsilon D} f(v)$= 2+2+2+2+2+1+1+1+1+1= 15

γ_{bd}=10

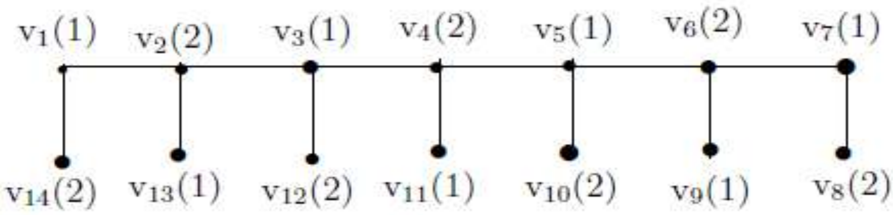

Figure 3

In this graph P_7 (n is odd) with pendant edge at each vertex, f (V) = 21, γ_{bd}=0

References

[1] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[2] B.Bresar, Tadeja Kraner Sumenjak, On the 2− rainbow domination in graphs, Discrete Applied Mathematics, 155(2007), 2394-2400.
[3] Henning, M. A. and S. T. Hedetniemi, Defending the Roman Empire–A new strategy, Discrete Mathematics 266, (2003), pp. 239–251.
[4] F. Haray, Graph Theory, Adison Wesley, reading Mass (1972).
[5] E.J. Cockayne, P.J.P. Grobler, W.R. Gründlingh, J. Munganga, and J.H. van Vuuren, Protection of a graph, Util. Math. 67 (2005) 19-32.
[6] S. Christilda and P. Namasivayam, The Balanced Domination Number of Some Standard Graphs, Proceedings on Recent Trends in Mathematical Sciences (2015) 92 – 96.
[7] D.B. West, Introduction to Graph Theory (Prentice-Hall, Inc, 2000).